RME 3204: System Transfer Function & Analysis

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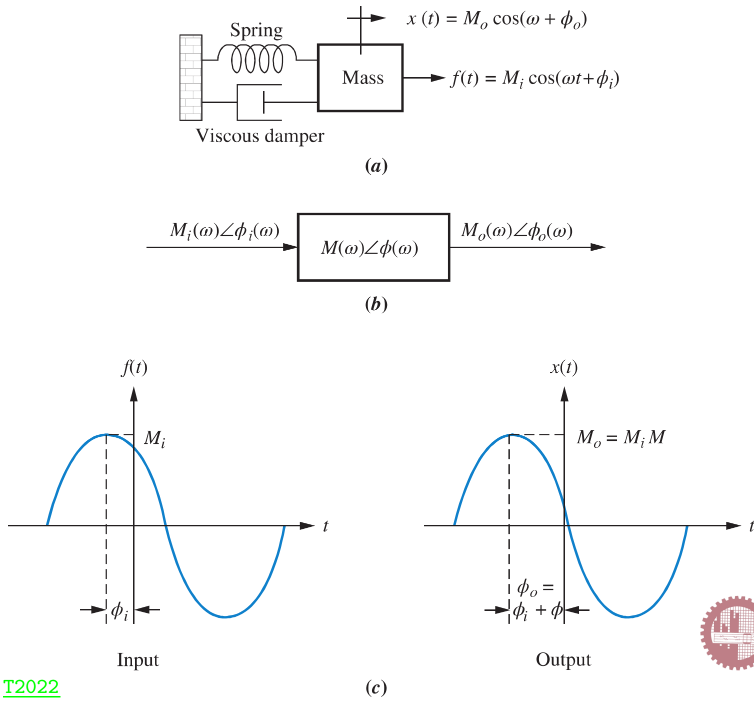
Transfer Function (TF)

$$\boxed{M_o(\omega)\angle\phi_o(\omega) = M_i(\omega)M(\omega)\angle[\phi_i(\omega)+\phi(\omega)]}$$

- Transfer function, $G(s) \equiv \frac{X(s)}{F(s)}$
- $F(s) \longrightarrow \boxed{G(s)} \longrightarrow X(s)$

$\Longrightarrow$ $\boxed{x(t) = f(t) \times G_a\angle\phi}$

- $G_a = |G(j\omega)|$, $\phi = \angle G(j\omega)$



T2022

Poles, Zeros, and System Response

- $G(s) = \frac{p(s)}{q(s)} = K\frac{(s+p_1)(s+p_2)\cdots(s+p_m)}{s^k(s+q_1)(s+q_2)\cdots(s+q_n)}$
- zeros of $G(s)$: values of s that make $p(s) = 0$

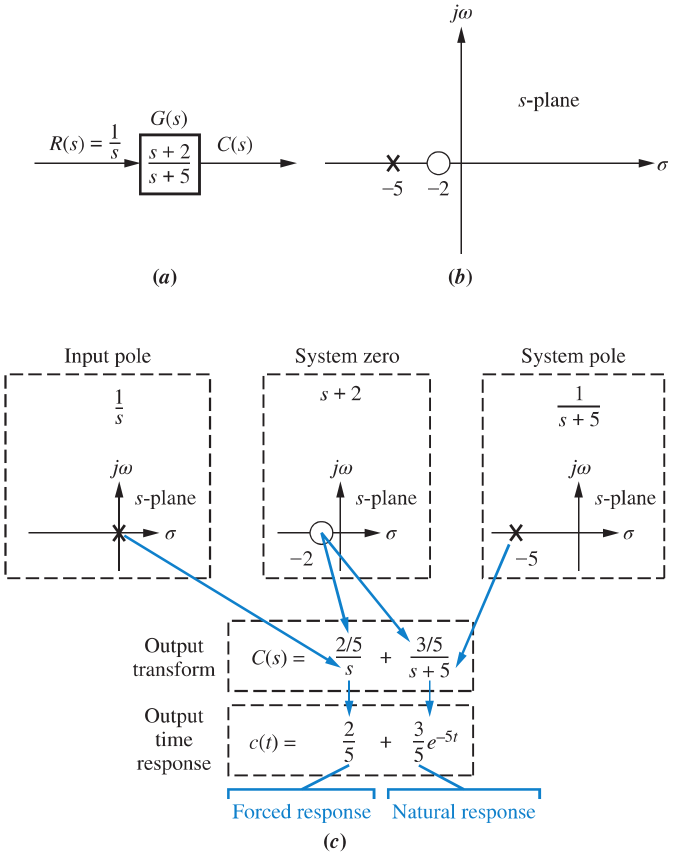
$\Rightarrow$ $s = -p_1, -p_2, \cdots, -p_m$.

- poles of $G(s)$: values of s that make $q(s) = 0$

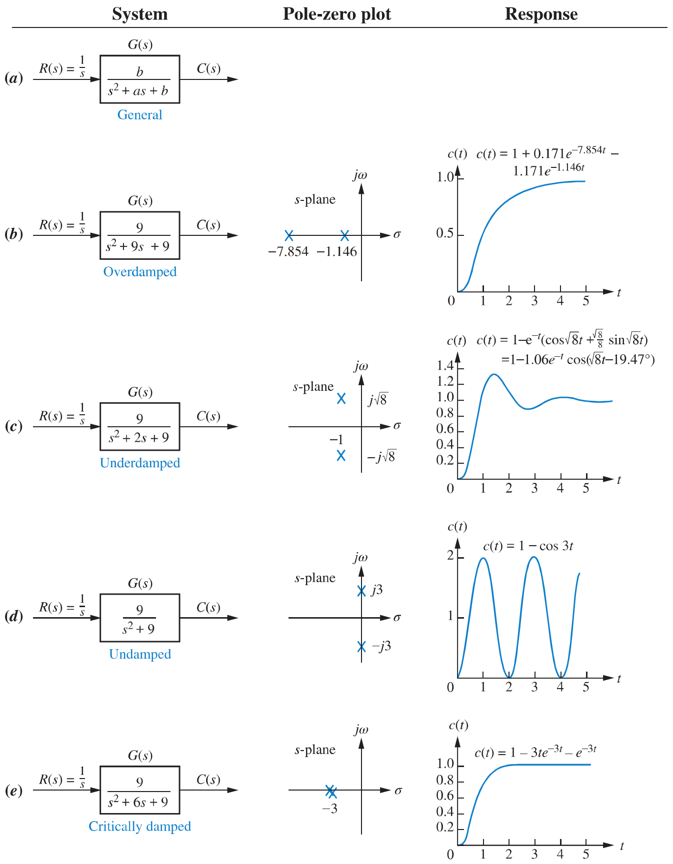
$\Rightarrow$ $s = -q_1, -q_2, \cdots, -q_n$ and k poles for s.

- Poles and zeros may be complex.

1. A pole of input function generates the form of forced response (steady-state response).
2. A pole of the TF generates the form of the natural response.
3. A pole on the real axis generates an exponential response of the form $e^{-\alpha t}$, where α is the pole location on the real axis.
4. zeros and poles generate amplitudes for forced and natural responses.


$R(s) = \frac{1}{s}$
$G(s)$
$\frac{s+2}{s+5}$
$C(s)$
(a)
$j\omega$
s-plane
σ
−5 −2
(b)
Input pole
System zero
System pole
Output transform
$C(s) = \frac{2/5}{s} + \frac{3/5}{s+5}$
Output time response
$c(t) = \frac{2}{5} + \frac{3}{5}e^{-5t}$
Forced response
Natural response
(c)

T2023


System
Pole-zero plot
Response
(a) General $G(s) = \frac{b}{s^2+as+b}$
(b) Overdamped $G(s) = \frac{9}{s^2+9s+9}$
−7.854 −1.146
$c(t) = 1 + 0.171e^{-7.854t} - 1.171e^{-1.146t}$
(c) Underdamped $G(s) = \frac{9}{s^2+2s+9}$
$j\sqrt{8}$ −1 $-j\sqrt{8}$
$c(t) = 1 - e^{-t}(\cos\sqrt{8}t + \frac{\sqrt{8}}{8}\sin\sqrt{8}t) = 1 - 1.06e^{-t}\cos(\sqrt{8}t - 19.47°)$
(d) Undamped $G(s) = \frac{9}{s^2+9}$
$j3$ $-j3$
$c(t) = 1 - \cos 3t$
(e) Critically damped $G(s) = \frac{9}{s^2+6s+9}$
−3
$c(t) = 1 - 3te^{-3t} - e^{-3t}$

T2024

Performance Parameters (Second-order System)

$$\boxed{\frac{1}{\omega_n^2}\frac{d^2x}{dt^2} + 2\frac{\zeta}{\omega_n}\frac{dx}{dt} + x = \Bbbk\, f(t) \Rightarrow G(s) = \frac{\omega_n^2/\Bbbk}{s^2+2\zeta\omega_n s+\omega_n^2}}$$

- Rise time, T_r: time required for the waveform to go from 0.1 of the final value to 0.9 of the final value.
- Peak time, T_p: time required to reach first, or maximum, peak.

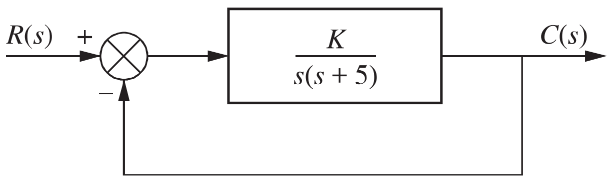
$\Rightarrow$ $\boxed{T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}}$

- Percent overshoot, %OS: amount that the waveform overshoots the steady-state, or final, value at the peak time, expressed as % of the steady-state value.

$\Rightarrow$ $\boxed{\%OS = \exp(-(\zeta\pi/\sqrt{1-\zeta^2})) \times 100}$

- Settling Time, T_s: is defined as the time for the response to reach, and stay within, 2% of its final value.

$\Rightarrow$ $\boxed{T_s = \frac{4}{\zeta\omega_n}}$



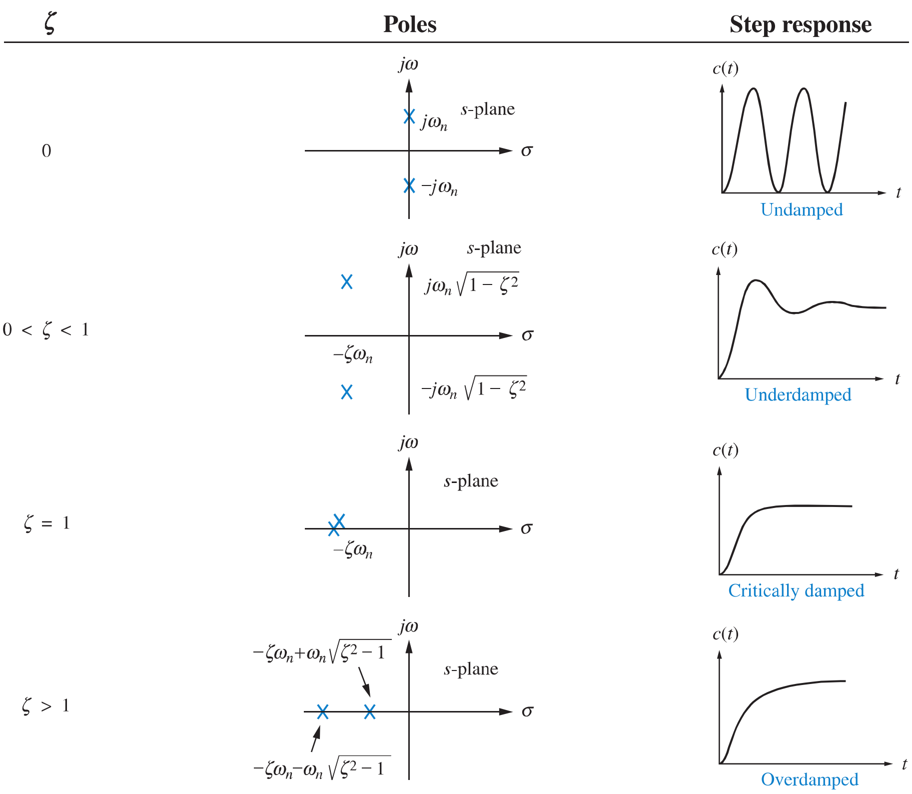
T2026

1. For $K = 25$, find the peak time, percent overshoot, and settling time.

[0.726 s, 16.3%, 1.6 s]

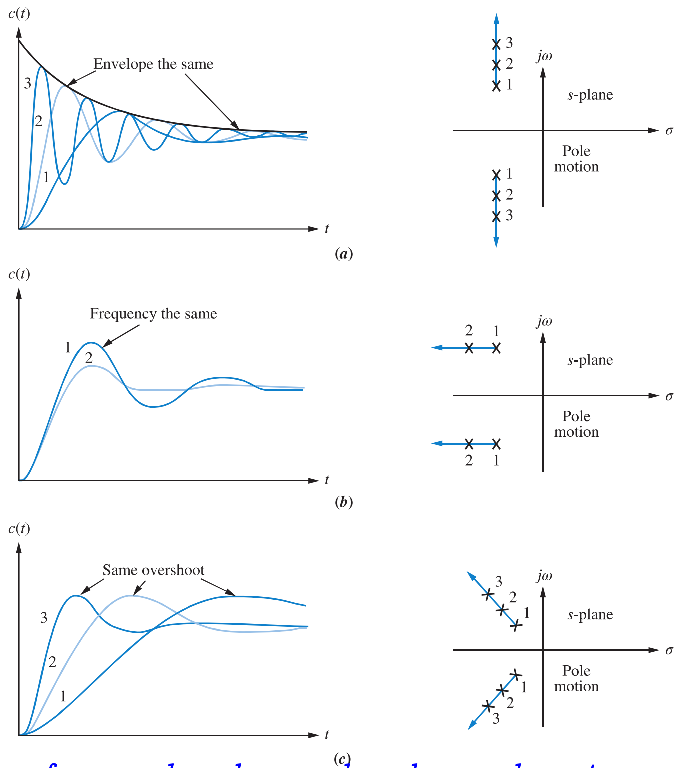
2. Design the value of gain, K so that the system will respond with a 10% overshoot.

[17.9]



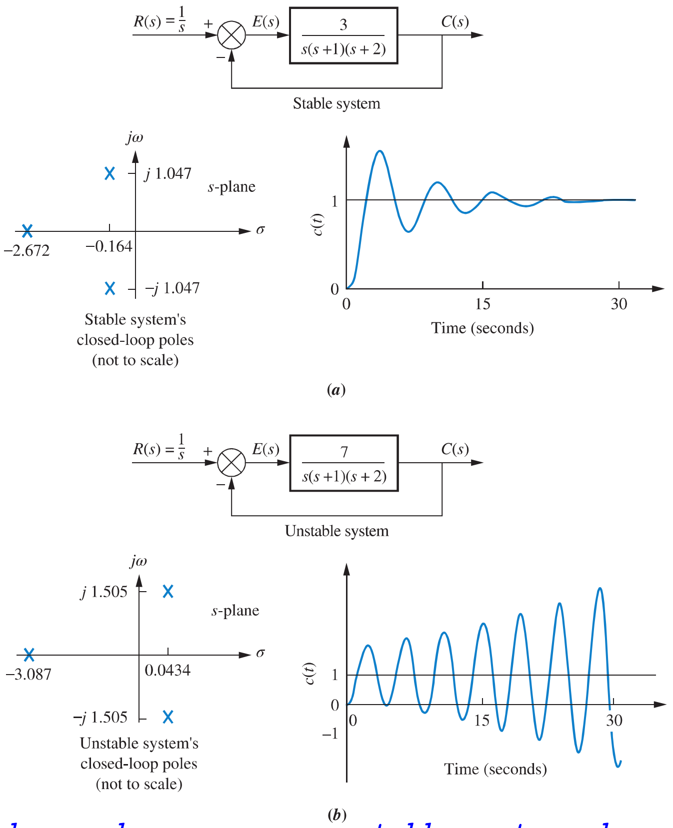
T2012

Second-order response as a function of damping ratio.



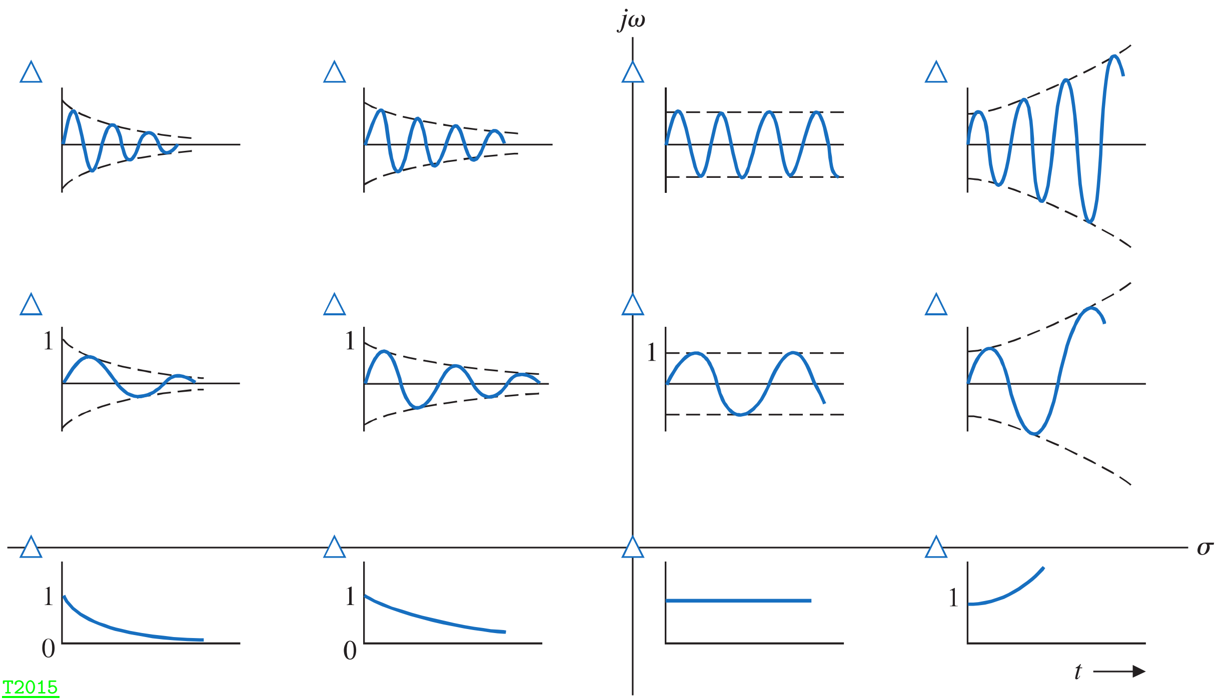
T2013

Step responses of second-order under-damped systems as poles move: with a. constant real part; b. constant imaginary part; c. constant damping ratio.



T2014

Closed-loop poles and response: a. stable system; b. unstable system

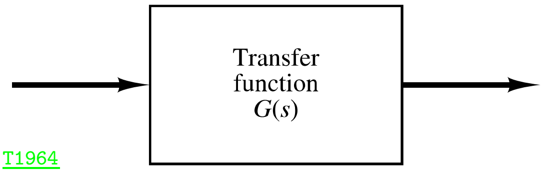


T2015

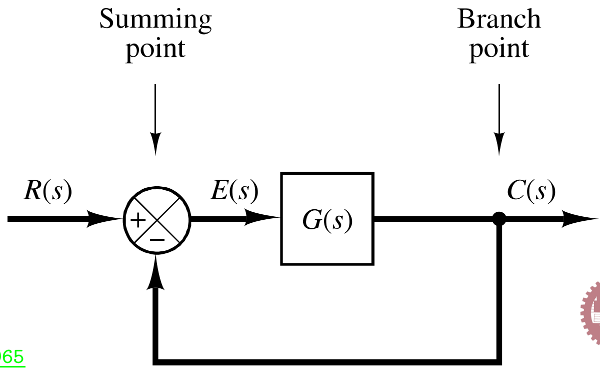
Impulse response for various root locations in the s-plane.

Block Diagrams

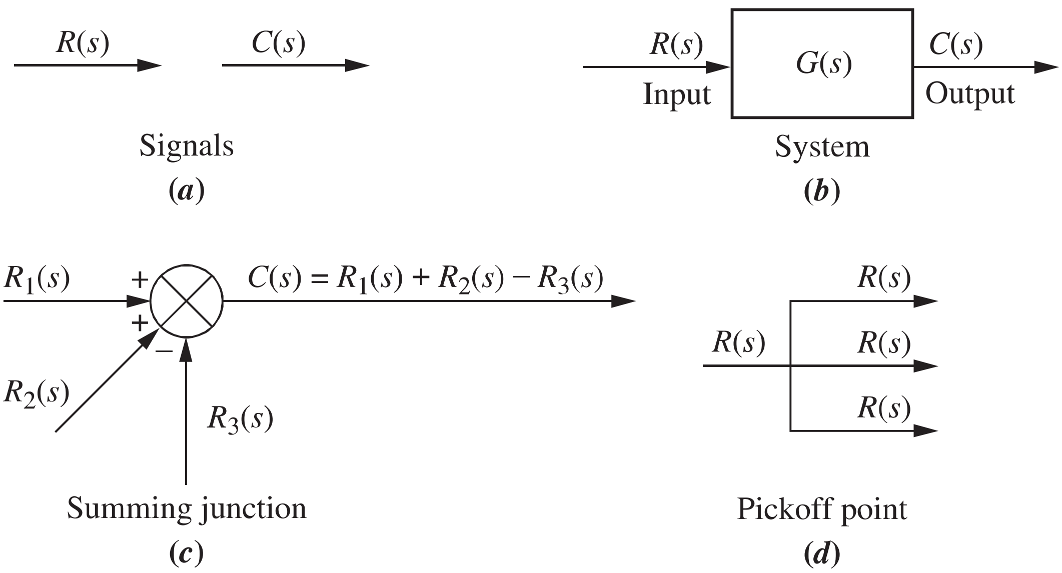
- Block Diagrams: A block diagram of a system is a pictorial representation of the functions performed by each component and of the flow of signals.
- Summing Point: A circle with a cross is the symbol that indicates a summing operation. The plus or minus sign at each arrowhead indicates whether that signal is to be added or subtracted. It is important that the quantities being added or subtracted have the same dimensions and the same units.
- Branch Point: A branch point is a point from which the signal from a block goes concurrently to other blocks or summing points.



T1964



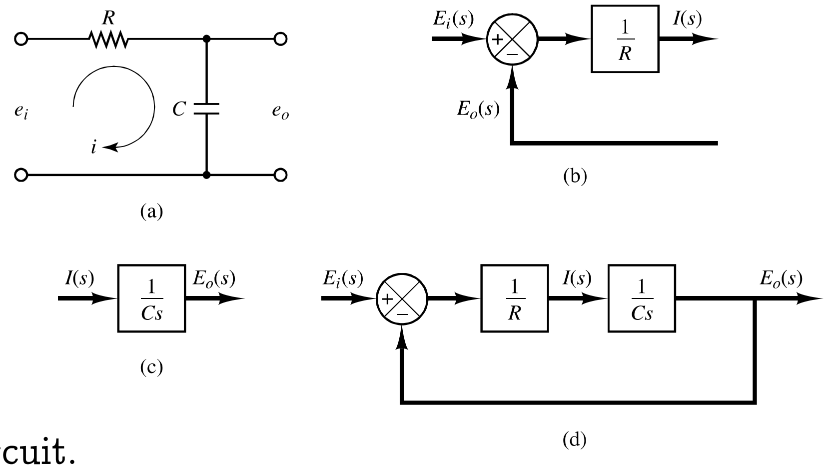
T1965



T1971

Procedures for Drawing a Block Diagram

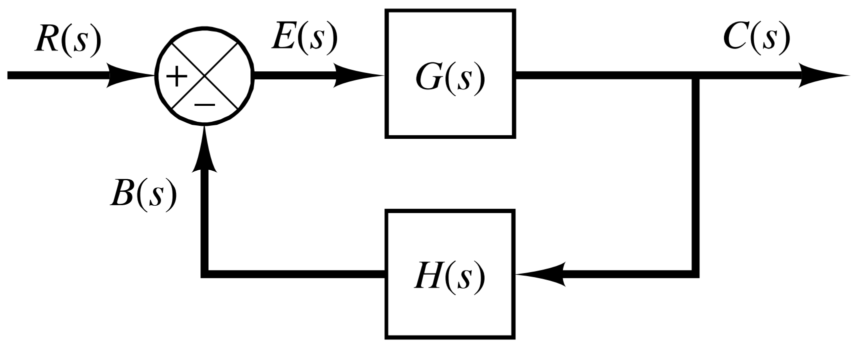
1. $i = \frac{e_i - e_o}{R} \rightarrow I(s) = \frac{E_i(s) - E_o(s)}{R}$
2. $e_o = \frac{\int i dt}{C} \rightarrow E_o(s) = \frac{I(s)}{Cs}$



T2009

(a) RC circuit.
(b) block diagram representing Equation (1);
(c) block diagram representing Equation (2);
(d) block diagram of the RC circuit.

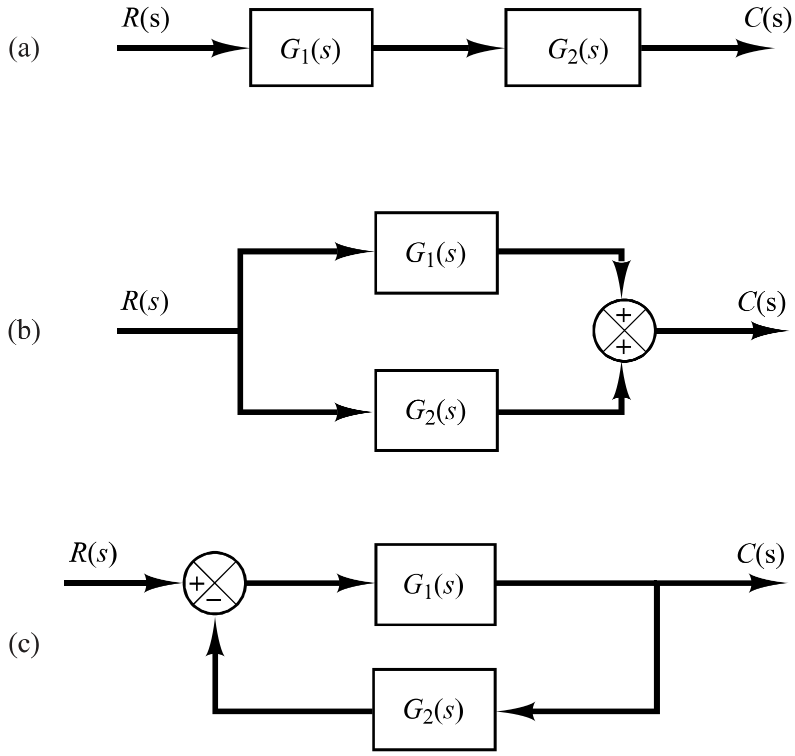
Transfer Function of Closed-loop System



T1966

- $C(s) = G(s)E(s)$
- $E(s) = R(s) - B(s) = R(s) - H(s)C(s)$

$\Longrightarrow \boxed{\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}} \Leftarrow$ closed-loop control transfer function.

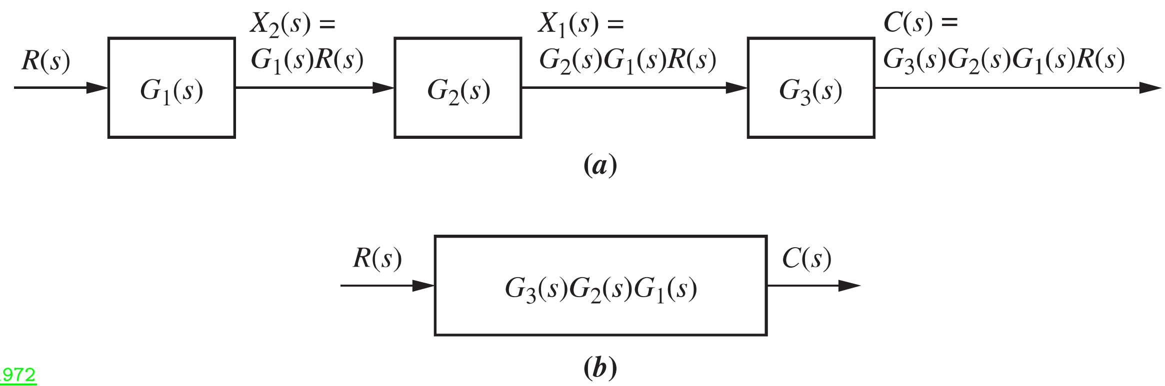


T1967

(a) Cascaded system; (b) parallel system; (c) feedback (closed-loop) system.

Combining blocks in cascade

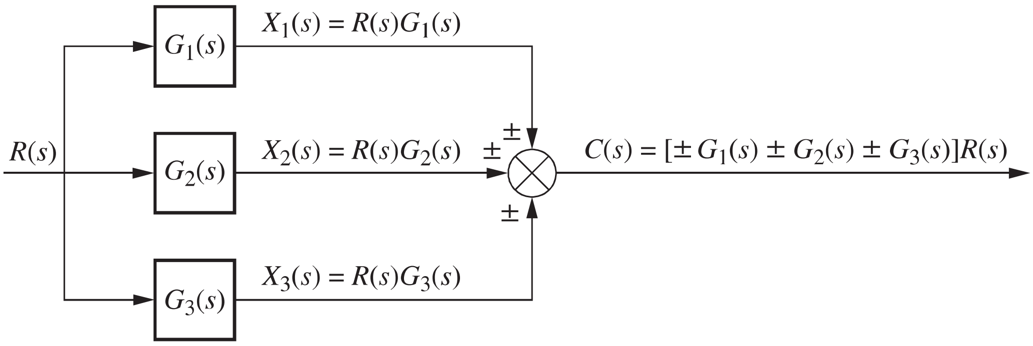
$$G_e(s) = \prod_{i=1}^{n} G_i$$



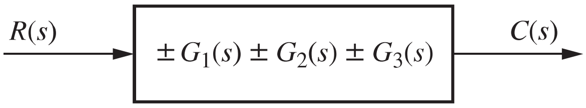
T1972

Combining blocks in parallel

$$\boxed{G_e(s) = \sum_{i=1}^{n} G_i}$$



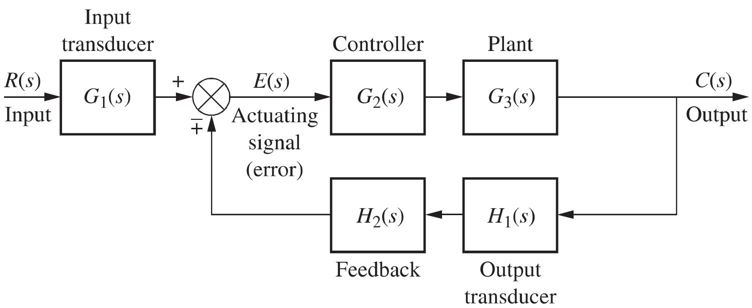
(a)



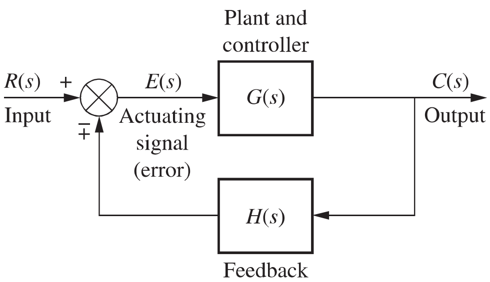
T1973

(b)

Feedback (closed-loop) System

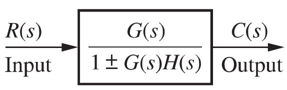


(a)

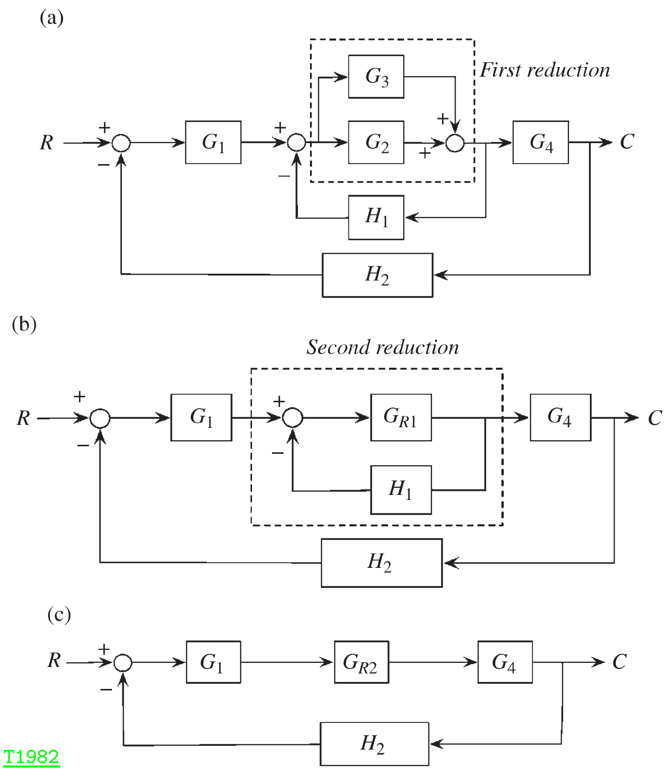


(b)

$$\boxed{G_e(s) = \frac{G(s)}{1 \pm G(s)H(s)}}$$



Example 01: Block diagram reduction

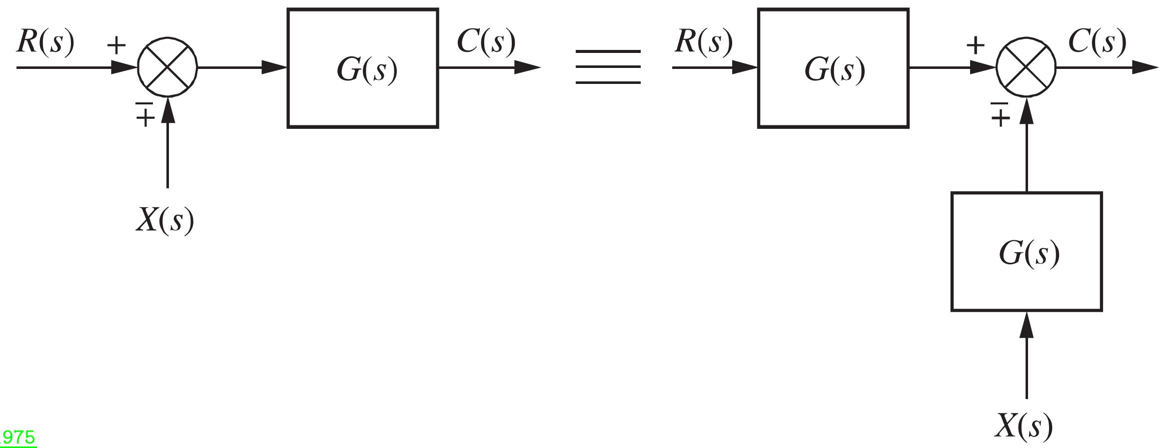


- $G_{R1} = G_2 + G_3$
- $G_{R2} = \frac{G_{R1}}{1+G_{R1}H_1}$
- $G = G_1 G_{R2} G_4$
- $\frac{C}{R} = \frac{G}{1+GH_2}$

$\Rightarrow \frac{C}{R} = \frac{G_1(G_2+G_3)G_4}{1+(G_2+G_3)H_1+G_1H_2(G_2+G_3)G_4}$

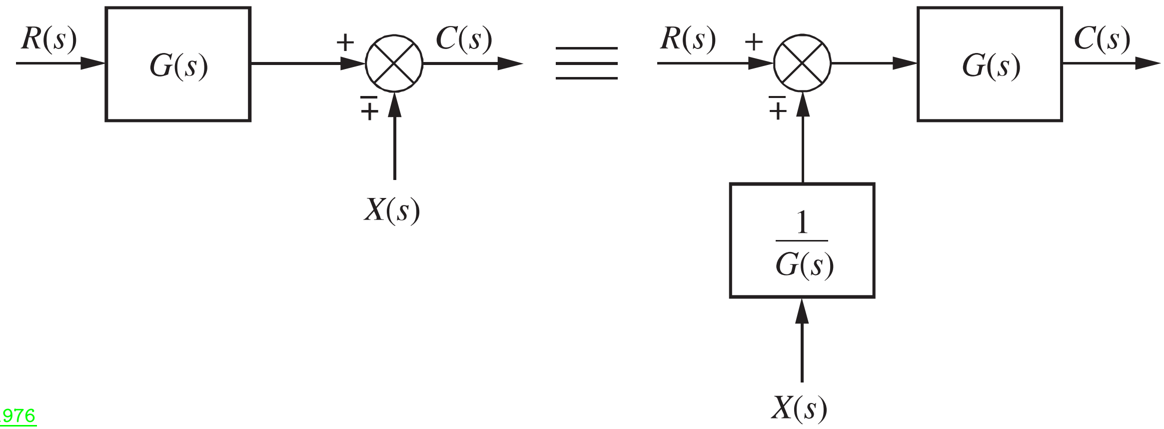
T1982

Moving a summing point behind a block



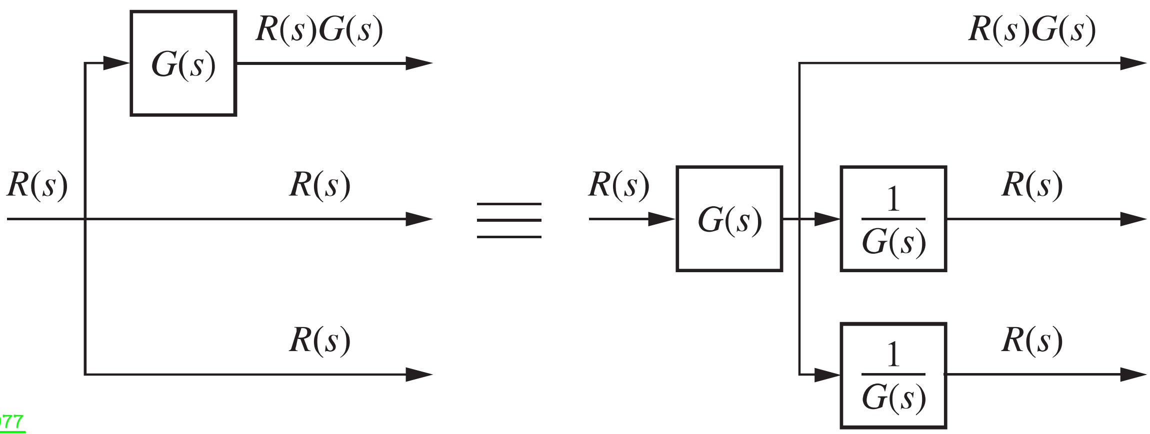
T1975

Moving a summing point ahead of a block



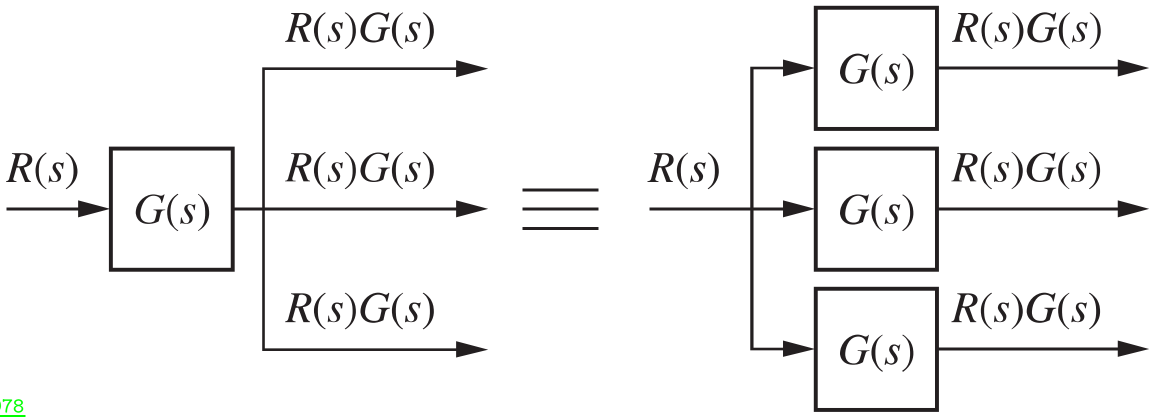
T1976

Moving a block to the left past a pickup point



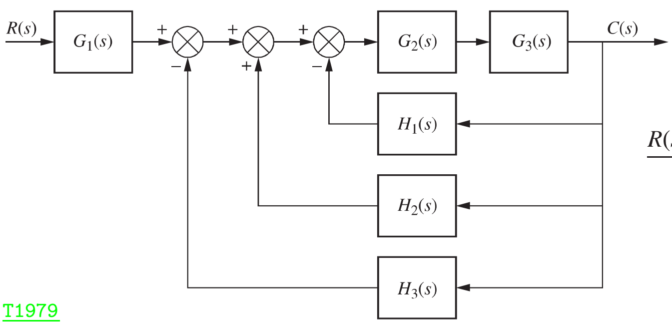
T1977

Moving a block to the right past a pickup point

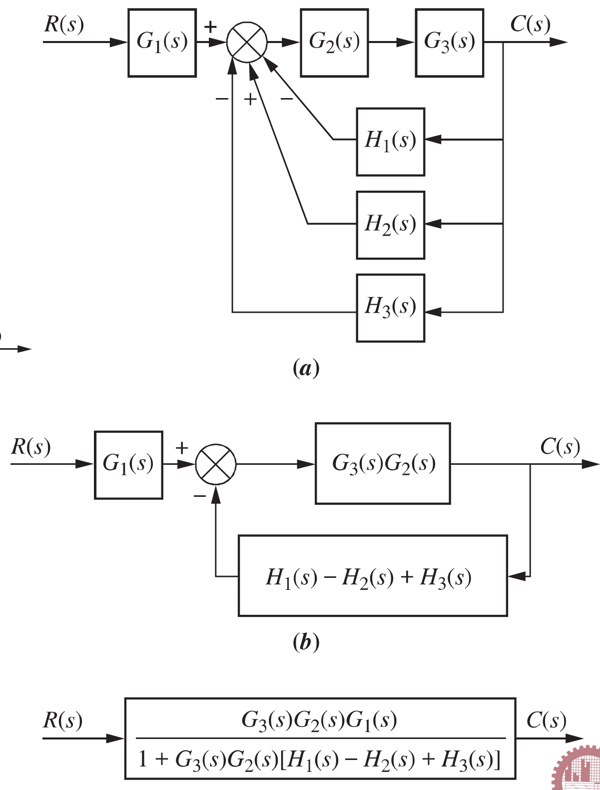


T1978

Example 02

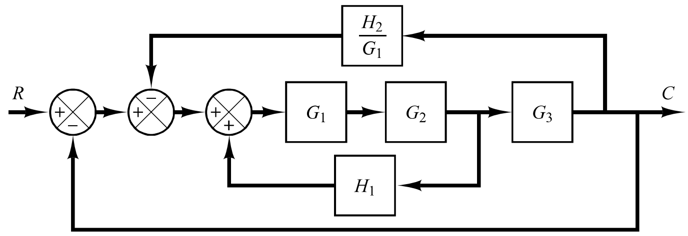


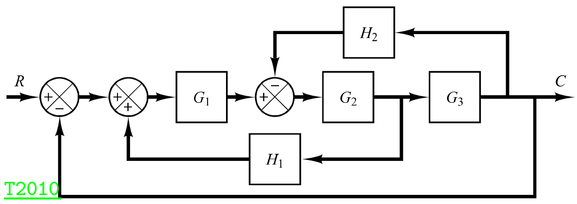
T1979



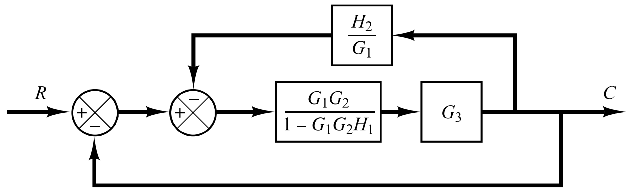
T1980

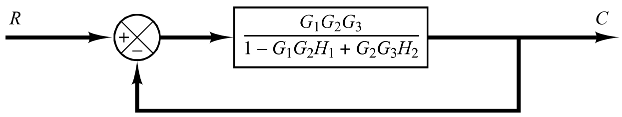
Example 03

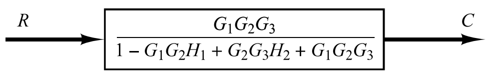




T2010

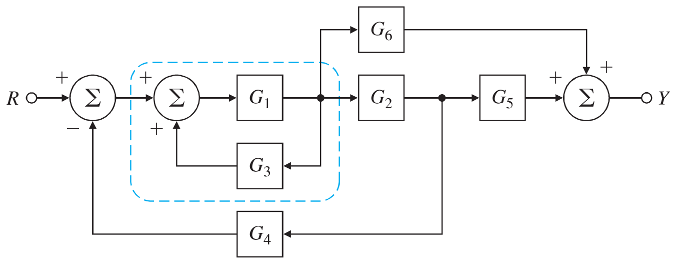




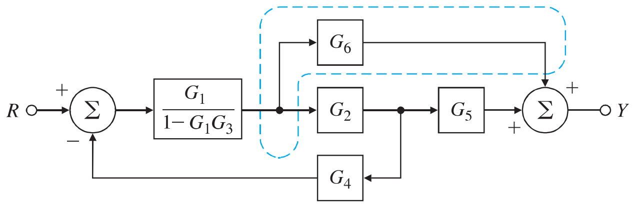


T2011

Example 04

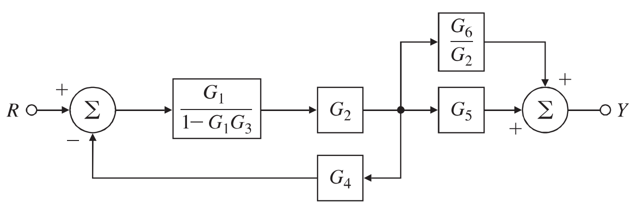


(a)

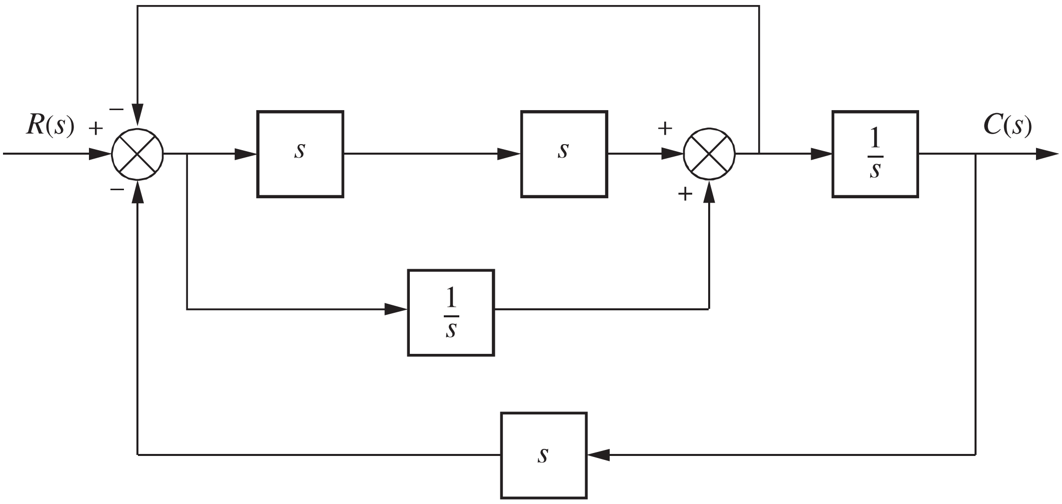


(b)

$$G(s) = \frac{G_1G_2G_5 + G_1G_6}{1 - G_1G_3 + G_1G_2G_4}$$



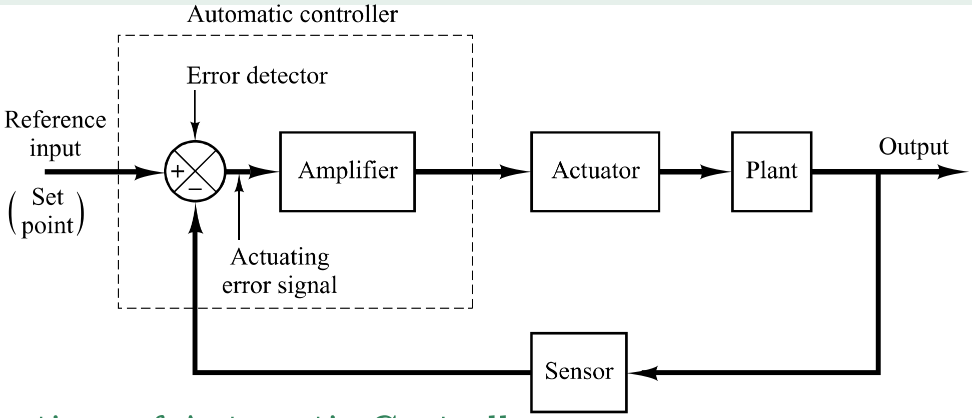
Example 05: Estimate equivalent transfer function



T1983

$$\Longrightarrow \frac{s^3+1}{2s^4+s^2+2s}$$

Automatic Controller



T1968

Classifications of Automatic Controllers

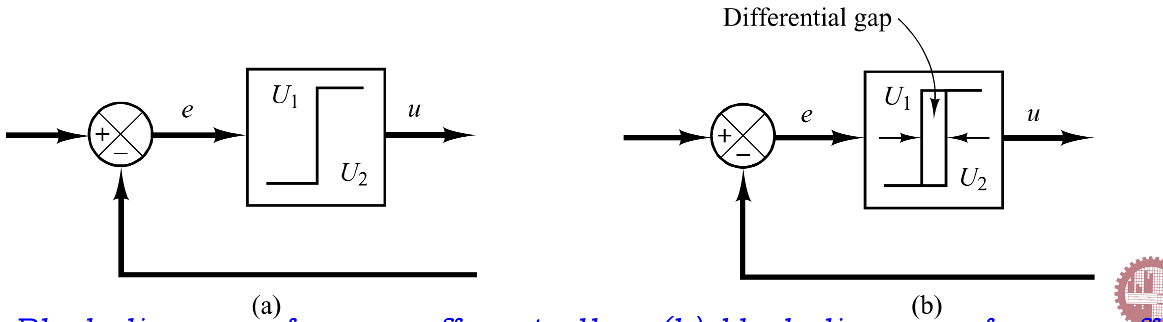
1. Two-position or on-off controllers
2. Proportional (P) controllers
3. Integral (I) controllers
4. Proportional-plus-integral (PI) controllers
5. Proportional-plus-derivative (PD) controllers
6. Proportional-plus-integral-plus-derivative (PID) controllers

Two-Position or On-Off Control Action

- The actuating element has only two fixed positions, simply on and off.
- If $u(t) \equiv$ controller output signal, $e(t) \equiv$ actuating error signal:

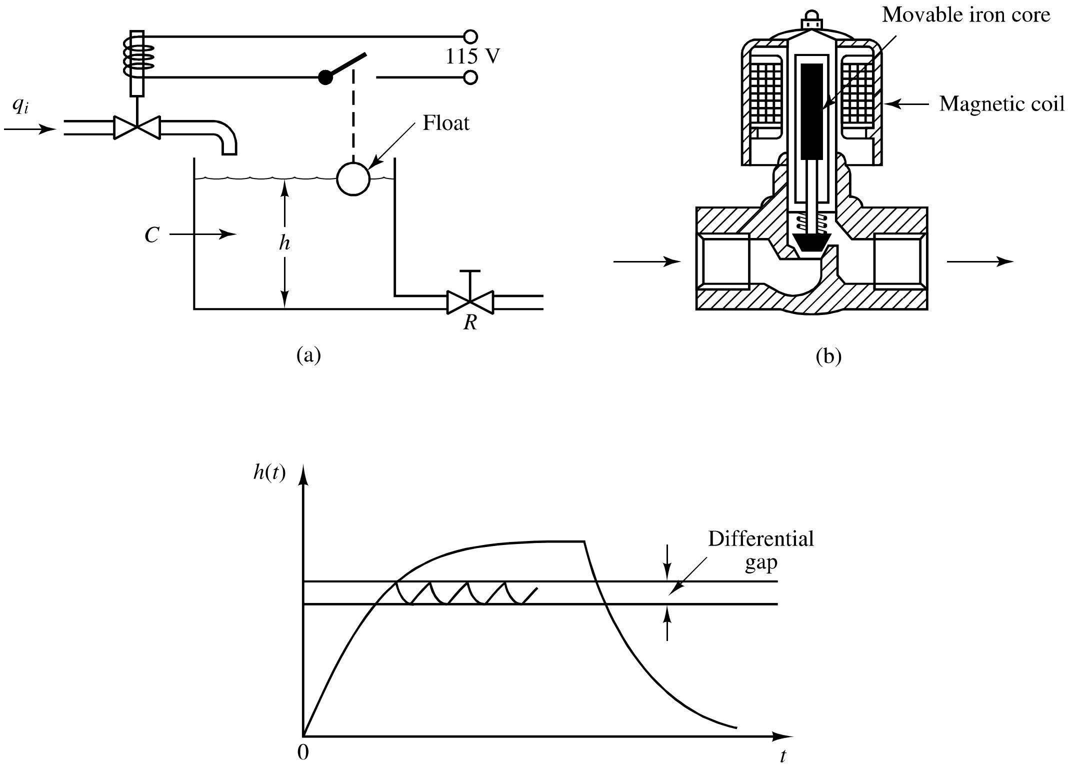
$$u(t) = \begin{cases} U_1 & \text{if } e(t) \geq 0, \\ U_2 & \text{if } e(t) < 0. \end{cases}$$

- Differential gaps are used to prevent too-frequent operation of the on-off mechanism.



T1969

(a) Block diagram of an on-off controller; (b) block diagram of an on-off



T1970

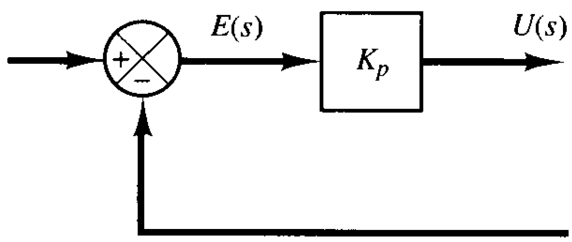
(a) Liquid-level control system; (b) electromagnetic valve. (c) Level h(t)-versus-time curve for the system.

Proportional (P) Controller

- The relationship between the output of the controller $u(t)$ and the actuating error signal $e(t)$ is:

$$u(t) = K_p\, e(t) :\longrightarrow \frac{U(s)}{E(s)} = K_p$$

- K_p is termed the proportional gain.
- P controller is essentially an amplifier with an adjustable gain.



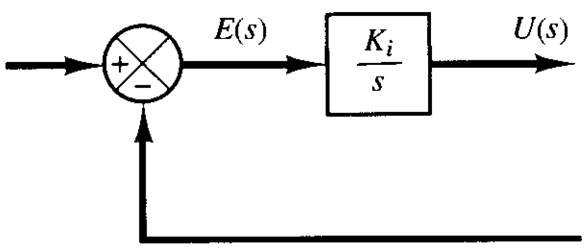
T2016

Integral (I) Controller

- The relationship between the output of the controller $u(t)$ and the actuating error signal $e(t)$ is:

$$u(t) = K_i \int_0^t e(t)\,dt :\longrightarrow \frac{U(s)}{E(s)} = \frac{K_i}{s}$$

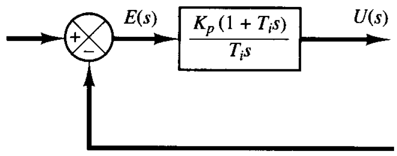
- K_i is an adjustable constant.
- For zero actuating error, the value of $u(t)$ remains stationary.
- Integral control action is sometimes called reset control.

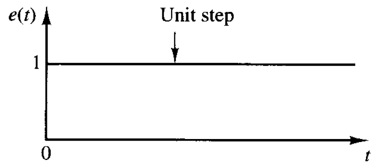


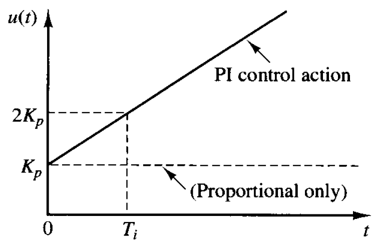
T2017

Proportional-Plus-Integral (PI) Controller

- $u(t) = K_p\, e(t) + \frac{K_p}{T_i}\int_0^t e(t)dt :\longrightarrow \frac{U(s)}{E(s)} = K_p\left(1 + \frac{1}{T_i s}\right)$
- K_p is termed the proportional gain, T_i is called integral time. Both K_p and T_i are adjustable.
- Inverse of T_i is called reset rate.



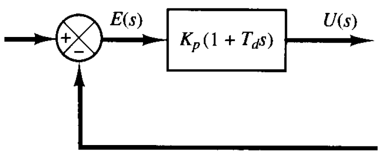


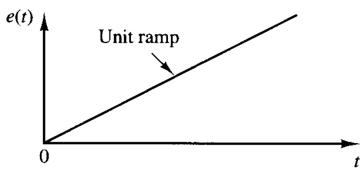


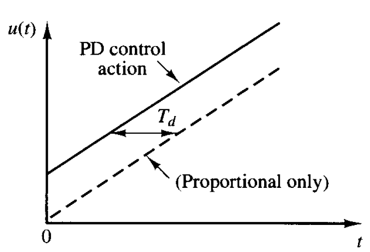
T2018

Proportional-Plus-Derivative (PD) Controller

- $u(t) = K_p\, e(t) + K_p T_d \frac{de(t)}{dt} :\longrightarrow \frac{U(s)}{E(s)} = K_p(1 + T_d s)$
- K_p is termed the proportional gain, T_d is called derivative time. Both K_p and T_d are adjustable.
- Derivative control action is sometimes called rate control.





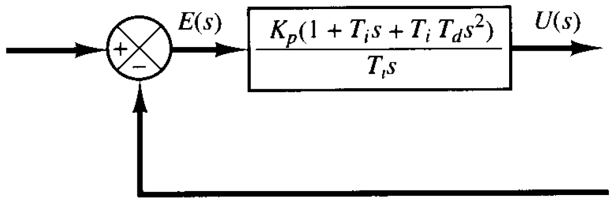


T2019

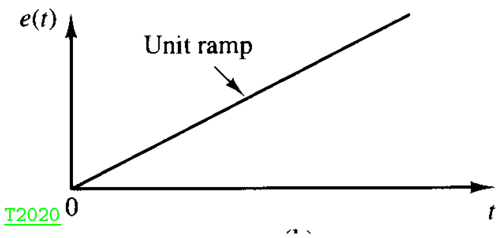
Proportional-Integral-Derivative (PID) Controller

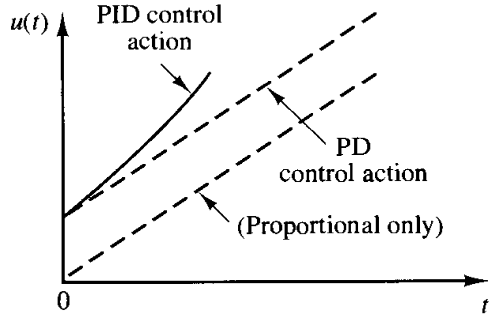
- $u(t) = K_p\, e(t) + \frac{K_p}{T_i}\int_0^t e(t)dt + K_p T_d \frac{de(t)}{dt}$

$\Rightarrow \frac{U(s)}{E(s)} = K_p\left(1 + \frac{1}{T_i s} + T_d s\right)$



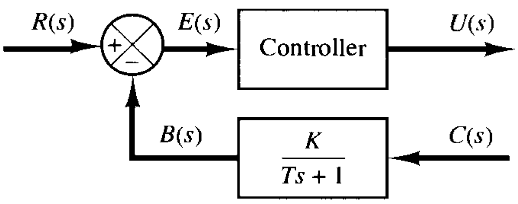
(a)



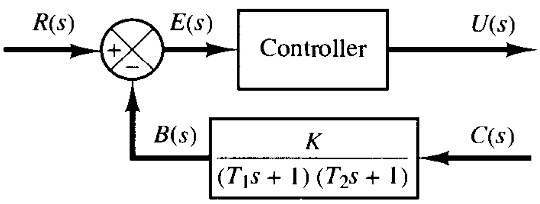


T2020

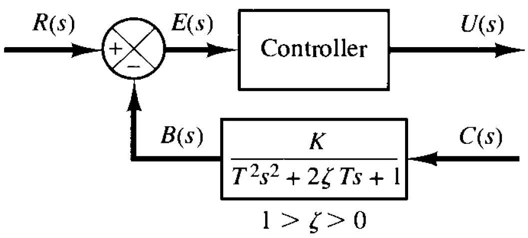
Effects of Sensors on System Performance



(a)



(b)

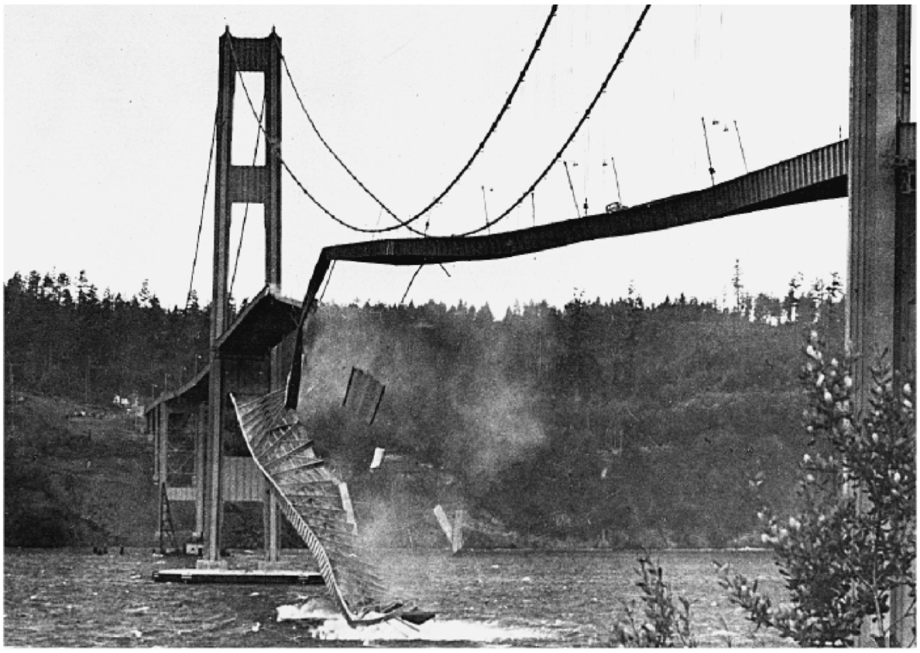


(c)

T2021

Block diagram of automatic controllers with (a) First-order sensor; (b) overdamped second-order sensor; (c) underdamped second-order system.

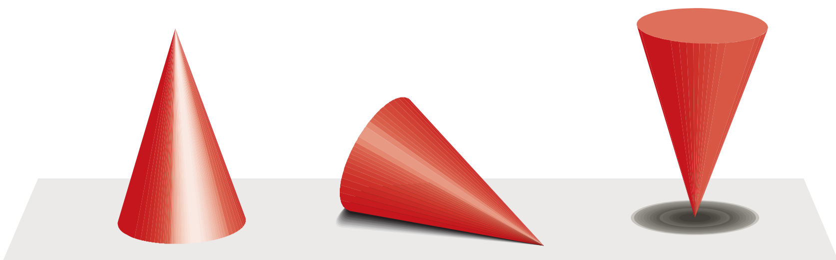
System Instability: Example



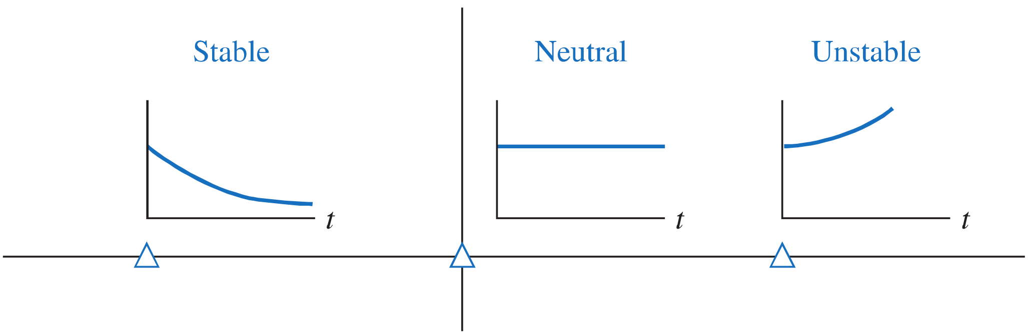
e133

Tacoma Narrows bridge at catastrophic failure

A stable system is a dynamic system with a bounded response to a bounded input.



(a) Stable (b) Neutral (c) Unstable



T1981

Routh-Hurwitz Criterion for Stability

The method involves two steps:

1. Generate a data table called a Routh table.
2. Interpret the Routh table to tell how many system-poles are in the left half-plane, the right half-plane, and on the $j\omega$-axis.

The Routh-Hurwitz Criterion declares that:
the number of roots of the polynomial that are on the right half-plane is equal to the number of sign changes of the first column of the Routh table.

Generation of a Basic Routh Table

$$R(s) \longrightarrow \boxed{\frac{N(s)}{a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}} \longrightarrow C(s)$$

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2			
s^1			
s^0			

e134

Initial layout for Routh table

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2	$\dfrac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1$	$\dfrac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$	$\dfrac{-\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0$
s^1	$\dfrac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1$	$\dfrac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$	$\dfrac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$
s^0	$\dfrac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1$	$\dfrac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$\dfrac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$

e135

Completed Routh table

Stability Analysis: R-W Criterion

Example: $R(s) \longrightarrow \boxed{\dfrac{1000}{s^3+10s^2+31s+1030}} \longrightarrow C(s)$

s^3	1	31	0
s^2	$\cancel{10}$ 1	$\cancel{1030}$ 103	0
s^1	$\dfrac{-\begin{vmatrix} 1 & 31 \\ 1 & 103 \end{vmatrix}}{1} = -72$	$\dfrac{-\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}}{1} = 0$	$\dfrac{-\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}}{1} = 0$
s^0	$\dfrac{-\begin{vmatrix} 1 & 103 \\ -72 & 0 \end{vmatrix}}{-72} = 103$	$\dfrac{-\begin{vmatrix} 1 & 0 \\ -72 & 0 \end{vmatrix}}{-72} = 0$	$\dfrac{-\begin{vmatrix} 1 & 0 \\ -72 & 0 \end{vmatrix}}{-72} = 0$

e136

Two sign changes in the first column, system is unstable since two poles exists in the right half-plane.

Special Cases of R-W Criterion

1. *Table have a zero only in the first column of a row.* In such case, an epsilon, ϵ, is assigned to replace the zero and the value of ϵ is then allowed to approach zero from either the positive or negative side after which the signs of the entries of the first column can be determined.
2. *Table have an entire row that consists of zeros.* In such case, we return to the row immediately above the row of zeros and form an auxiliary polynomial, using the entries in that row as coefficients. The polynomial is then differentiated with respect to s to obtain the coefficients to replace the rows of zeros.

Example of a R-W Criterion: Case 1

$$R(s) \longrightarrow \boxed{\frac{10}{s^5+2s^4+3s^3+6s^2+5s+3}} \longrightarrow C(s)$$

s^5	1	3	5
s^4	2	6	3
s^3	$\cancel{0}\ \epsilon$	$\frac{7}{2}$	0
s^2	$\frac{6\epsilon - 7}{\epsilon}$	3	0
s^1	$\frac{42\epsilon - 49 - 6\epsilon^2}{12\epsilon - 14}$	0	0
s^0	3	0	0

e137

Label	First Column	$\epsilon = +$	$\epsilon = -$
s^5	1	$+$	$+$
s^4	2	$+$	$+$
s^3	$\cancel{0}\ \epsilon$	$+$	$-$
s^2	$\frac{6\epsilon - 7}{\epsilon}$	$-$	$+$
s^1	$\frac{42\epsilon - 49 - 6\epsilon^2}{12\epsilon - 14}$	$+$	$+$
s^0	3	$+$	$+$

e138

The system is unstable, with two poles in the right half-plane

Example of a R-W Criterion: Case 2

$$R(s) \longrightarrow \boxed{\frac{10}{s^5+7s^4+6s^3+42s^2+8s+56}} \longrightarrow C(s)$$

s^5	1	6	8
s^4	$\cancel{7}\ 1$	$\cancel{42}\ 6$	$\cancel{56}\ 8$
s^3	$\cancel{0}\ \cancel{4}\ 1$	$\cancel{0}\ \cancel{12}\ 3$	$\cancel{0}\ \cancel{0}\ 0$
s^2	3	8	0
s^1	$\frac{1}{3}$	0	0
s^0	8	0	0

e139

No sign change, stable system

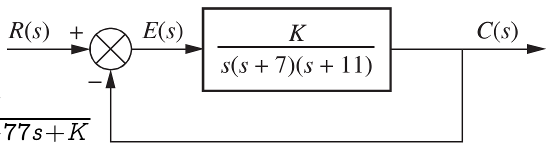
Case 2: Procedure Steps

1. Start by forming Routh table.
2. At 2^{nd} row, multiply through 1/7 for convenience.
3. At 3^{rd} row, entire row consists of zeros; then
   - return to the row immediately above the row consisting zeros,
   - form an auxiliary polynomial, using the entries in that row,

$$P(s) = s^4 + 6s^2 + 8 \Longrightarrow \frac{dP(s)}{ds} = 4s^3 + 12s + 0$$

   - the coefficients [4,12,0] are then used to replace the zeros,
   - for convenience, the 3^{rd} row is multiplied by 1/4
4. Remain of the table is formed in standard manner.

Example: ▷ Find the range of gain, K for the system that will cause the system to be stable, unstable, and marginally stable. Assume $K > 0$.



T2027 $G(s) = \frac{K}{s^3+18^2+77s+K}$

s^3	1	77
s^2	18	K
s^1	$\frac{1386-K}{18}$	
s^0	K	

T2028

- For $K < 1386$: stable system.
- For $K > 1386$: unstable system.
- For $K = 1386$: entire row of zeros:

$$P(s) = 18s^2 + 1386 \Longrightarrow \frac{dP(s)}{ds} = 36s + 0$$

$\Rightarrow$ Column 1: [1,18,36,1386], no sign change, stable system.