RME 3204: Dynamic Physical Models & Responses

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Basic System Models

- Modelling is the process of representing the behaviour of a system by a collection of mathematical equations and logics. It is comprehensively utilized to study the response of any system.
- Response of a system is a measure of its fidelity to its purpose.
- Simulation is the process of solving the model and it is performed using computer(s).
- Equations are used to describe the relationship between the input and output of a system.

$$\textit{Input} \Longrightarrow \boxed{\text{Governing Equations}} \Longrightarrow \textit{Output}$$

- Analogy approach is widely used to study system response.

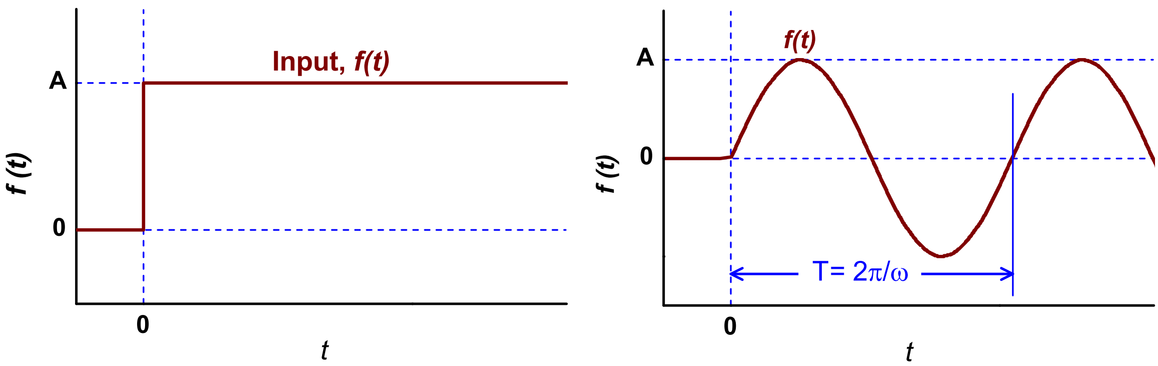
System Response

Response is a measure of a system's fidelity to purpose.

1. Amplitude response:
   - A linear response to various input amplitudes within range.
   - Beyond the linear range, the system is said to be over-driven.
2. Frequency response: is the ability of the system to treat all frequencies the same so that the gain amplitude remains the same over the frequency range desired.
3. Phase response: is important for complex waveforms. Lack of good response may result in severe distortion.
4. Delay, Rise time, Slew rate:
   - Delay or rise time is required to respond to an input quantity.
   - *Slew rate* is the maximum applicable rate of change.

Step and Harmonic Inputs

- step function: $f(t) = \begin{cases} 0 & \text{at} \quad t \le 0 \\ A & \text{for} \quad t > 0 \end{cases}$
- harmonic function: $f(t) = \begin{cases} 0 & \text{at} \quad t \le 0 \\ A \sin \omega t & \text{for} \quad t > 0 \end{cases}$



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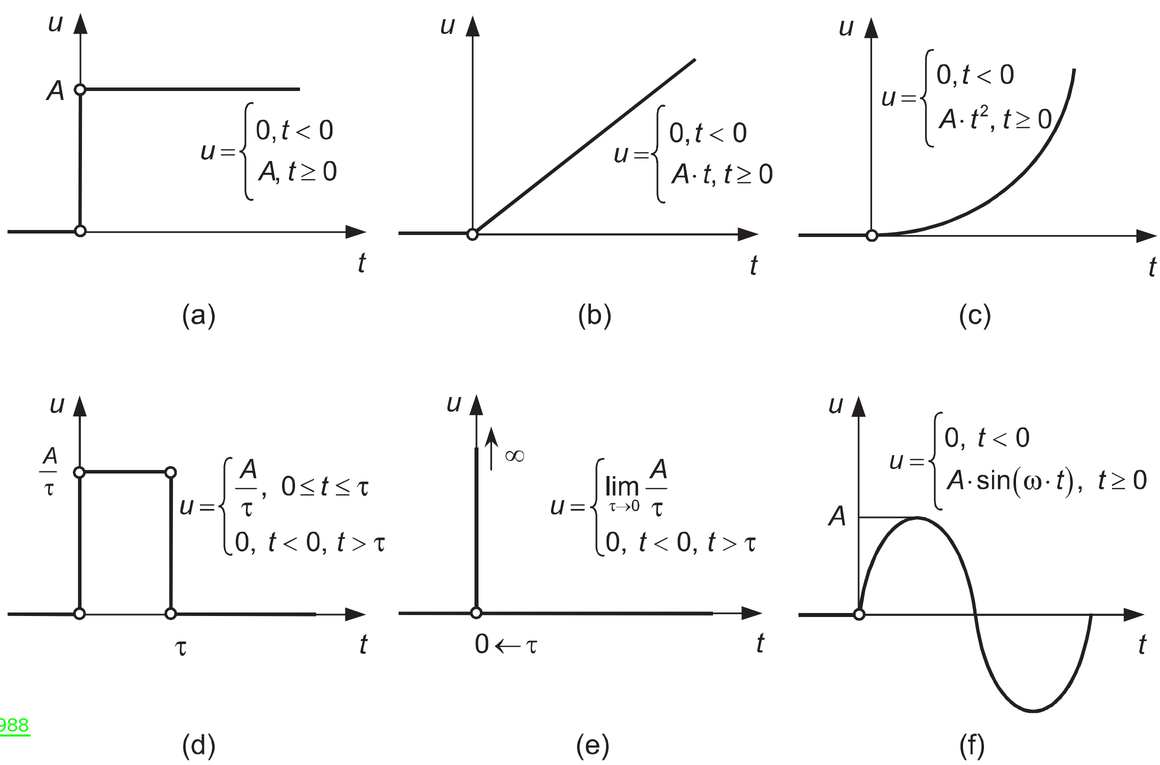
(a) Step input (b) Harmonic input

Step and harmonic inputs are widely used to analyse system response.



T1988

A Few Input Functions: (a) Step; (b) Ramp; (c) Parabolic; (d) Pulse; (e) Impulse; (f) Sinusoidal.

Bode Diagram

- Frequency response of a system is described by the set of values of gain (G_a) and phase angle (ϕ) when a sinusoidal input is varied over a range of frequencies (ω).
- Bode diagram is a pair of graphs which consists of two plots:
  1. Logarithmic gain, $L(\omega) \equiv 20 \log_{10} G_a(\omega)$ vs. $\log_{10}(\omega)$, and
  2. Phase angle, $\phi(\omega)$ vs. $\log_{10}(\omega)$.
- The vertical scale of the amplitude Bode diagram is in decibels (dB), where a non-dimensional frequency parameter such as frequency ratio, (ω/ω_n), is often used on the horizontal axis.

ω	$\log_{10}(\omega)$	$20 \log_{10} \lvert G(j\omega) \rvert$ in dB
0	$-\infty$	$20 \log_{10} \lvert 1/1 \rvert = 0$
$0.1/\tau$	$-1 + \log_{10}(1/\tau)$	$20 \log_{10} \left\lvert \frac{1}{0.1j+1} \right\rvert \approx 0$
$1/\tau$	$\log_{10}(1/\tau)$	$20 \log_{10} \left\lvert \frac{1}{j+1} \right\rvert = 20 \log_{10} \lvert 1/\sqrt{2} \rvert = -3$
$10/\tau$	$1 + \log_{10}(1/\tau)$	$20 \log_{10} \left\lvert \frac{1}{10j+1} \right\rvert = 20 \log_{10} \lvert 1/\sqrt{101} \rvert \approx -20$
$100/\tau$	$2 + \log_{10}(1/\tau)$	$20 \log_{10} \left\lvert \frac{1}{100j+1} \right\rvert \approx 20 \log_{10} \lvert 1/100 \rvert = -40$
$1000/\tau$	$3 + \log_{10}(1/\tau)$	$20 \log_{10} \left\lvert \frac{1}{1000j+1} \right\rvert \approx 20 \log_{10} \lvert 1/1000 \rvert = -60$
∞	∞	$20 \log_{10} \left\lvert \frac{1}{\infty j+1} \right\rvert = 20 \log_{10} \lvert 0 \rvert = -\infty$

T1994

ω	$\log_{10}(\omega)$	$\angle G(j\omega) = \angle \frac{1}{\tau j\omega + 1} = -\tan^{-1}\left(\frac{\omega}{1/\tau}\right)$
0	$-\infty$	$-\tan^{-1}(0) = 0°$
$(0.1)(1/\tau)$	$-1 + \log_{10}(1/\tau)$	$-\tan^{-1}(0.1) = -5.7°$
$1/\tau$	$\log_{10}(1/\tau)$	$-\tan^{-1}(1) = -45°$
$10(1/\tau)$	$1 + \log_{10}(1/\tau)$	$-\tan^{-1}(10) = -84.3°$
$100(1/\tau)$	$2 + \log_{10}(1/\tau)$	$-\tan^{-1}(100) \approx -90°$

T1995

Transfer Function (TF)

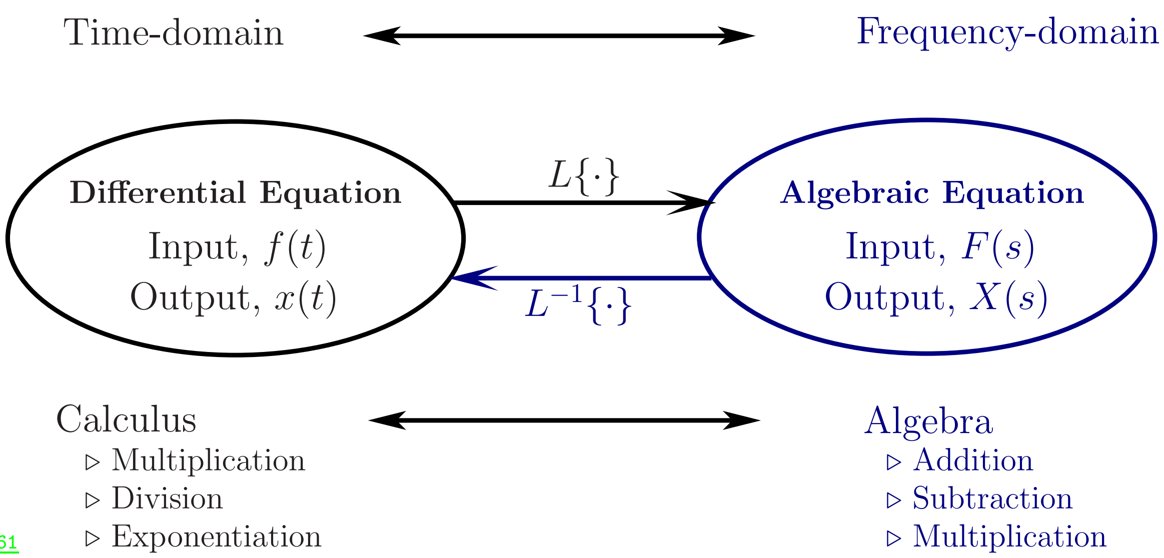
- Transfer function of a linear system, $G(s)$, is defined as the ratio of the Laplace transform (LT) of the output variable, $X(s) \equiv \mathcal{L}\{x(t)\}$, to the LT of the input variable, $F(s) \equiv \mathcal{L}\{f(t)\}$, *with all the initial conditions are assumed to be zero.*

$$\boxed{G(s) \equiv \frac{X(s)}{F(s)}}$$

○ The Laplace operator, $s \equiv \sigma + j\omega$, is a complex variable. For steady-state sinusoidal input, $\sigma = 0$, and system response can be evaluated by setting $s = j\omega$.

- ○ Amplitude gain, $G_a(\omega) \equiv |G(j\omega)|$
- ○ Phase lag, $\phi(\omega) \equiv \angle G(j\omega)$

$$F(s) \longrightarrow \boxed{G(s)} \longrightarrow X(s) \quad :\Longrightarrow \quad x(t) = f(t) \times G_a \angle \phi$$



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Important Laplace Transform Pairs

$f(t)$	$F(s)$
$\delta(t)$	1
t^n	$\frac{n}{s^{n+1}}$
Step function, A	A/s
e^{-at}	$\frac{1}{s+a}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
$\frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin \omega_n \sqrt{1-\zeta^2}\ t,\ \zeta < 1$	$\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$

Modelling of a General System

The response of a system, i.e., output, $x(t)$, when subjected to an input forcing function, $f(t)$, may be expressed by a linear ordinary differential equation with constant coefficients of the form:

$$a_n \frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \cdots + \underbrace{a_2 \frac{d^2 x}{dt^2} + \underbrace{a_1 \frac{dx}{dt} + \overbrace{a_0 x = f(t)}^{0^{th}\ order}}_{1^{st}\ order}}_{2^{nd}\ order}$$

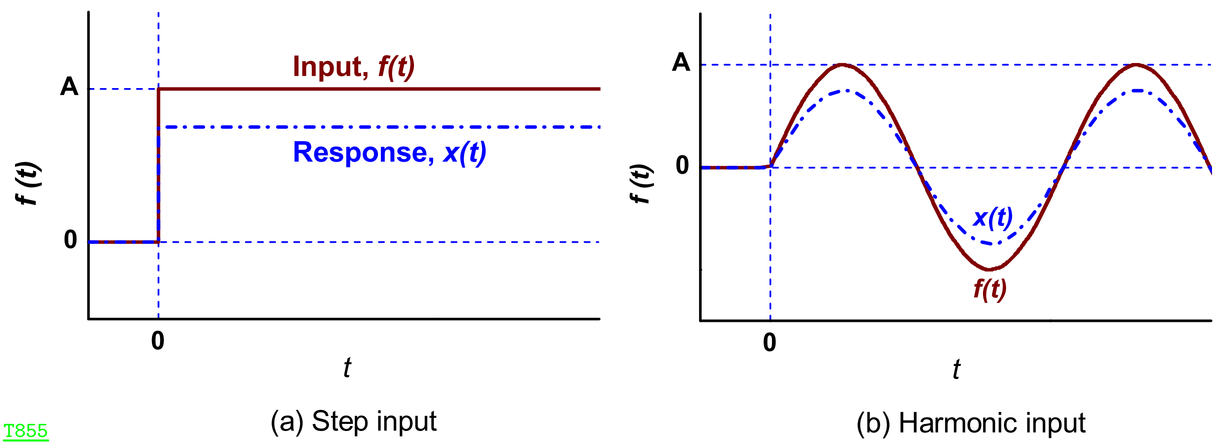
$f(t)$	$\equiv$	Input quantity imposed on the system,
$x(t)$	$\equiv$	Output or the response of the system,
a's	$\equiv$	Physical system parameters, assumed constants.

↬ Order of a system is designated by the order of the DE.

Zeroth Order System

$$\boxed{a_0 x = f(t) :\Longrightarrow x(t) = \Bbbk\, f(t)}$$

- $\Bbbk \equiv \frac{1}{a_o} \equiv$ Static sensitivity or gain: the scaling factor between the input and the output. For any-order system, it always has the same physical interpretation, i.e., the amount of output per unit input when the input is static and under such condition all the derivative terms of general equation are zero.
- No equilibrium seeking force is present.
- Output follows the input without distortion or time lag.
- System requires no additional dynamic considerations.
- Represents ideal dynamic performance.
- Example: Potentiometer, ideal spring etc.



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Zero-order instrument's response for step and harmonic inputs (for $\Bbbk = 0.75$).

First Order System

$$\boxed{a_1 \frac{dx}{dt} + a_0\, x = f(t) :\Longrightarrow \tau\, \frac{dx}{dt} + x = \Bbbk\, f(t)}$$

$$\begin{aligned} \Bbbk &\equiv 1/a_o &&\equiv \text{static sensitivity,} \\ \tau &\equiv a_1/a_o &&\equiv \text{time-constant.} \end{aligned}$$

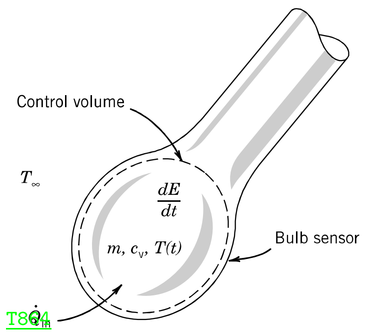
$a_o \iff$ dissipation (electric or thermal resistance).

$a_1 \iff$ storage (electric or thermal capacitance).

$\looparrowright$ Example: Thermometer, capacitor etc.

- Time constant, τ has the dimension of time, while the *static sensitivity*, $\Bbbk$ has the dimension of output divided by input.
- When $\tau \to 0$: the effect of the derivative terms becomes negligible and the governing equation approaches to that of a zero-order system.

▶ Consider a thermocouple initially at temperature, T is suddenly exposed to an environment at T_∞.



- $h \equiv$ convective heat transfer coefficient,
- $A \equiv$ heat transfer surface area,
- $m \equiv$ mass of mercury + bulb,
- $C \equiv$ specific heat of mercury + bulb.

$$\dot{Q}_{in} = hA\,[T_\infty - T(t)] = mC\frac{dT(t)}{dt}$$

$$\boxed{\tau \frac{dT(t)}{dt} + T(t) = T_\infty}$$

- Time constant, $\tau \equiv \frac{mC}{hA}$
- Static sensitivity, $\Bbbk = 1.0$
- $m \uparrow C \uparrow \;\; h \downarrow A \downarrow \Longrightarrow \tau \uparrow$
- Systems with small $\tau \rightsquigarrow$ good dynamic response.

Response of a 1^{st} Order System: Step Input

$$\boxed{x = x_o,\ f = 0 : t = 0; \qquad f(t) = A : t > 0}$$

$$\tau \frac{dx}{dt} + x = \mathbb{k}\, f(t)$$

$$\implies x(t) = \underbrace{(x_o - A\mathbb{k})\exp(-t/\tau)}_{\text{transient response}} + \overbrace{A\mathbb{k}}^{\text{steady-state response}}$$

- $x(t \to \infty) = A\mathbb{k} = x_\infty \Longleftarrow$ Steady-state Response
- Error, $e_m = x_\infty - x(t) = (x_\infty - x_o)e^{-t/\tau}$
- Non-dimensional Error, $e_m/(x_\infty - x_o) = e^{-t/\tau}$

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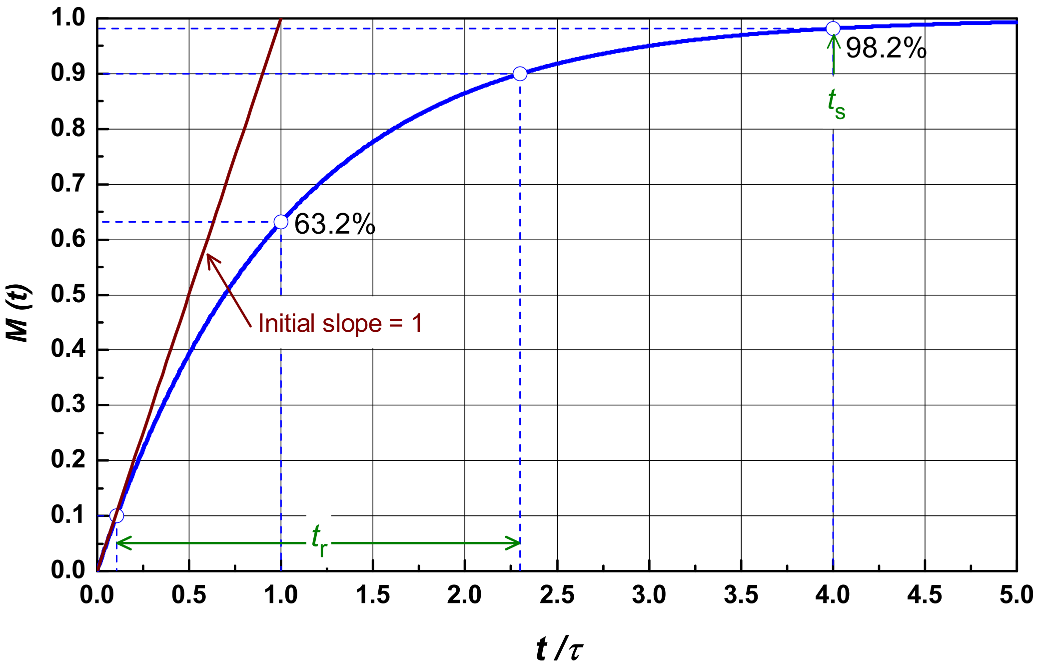
- Non-dimensional response, $M(t) = \frac{x(t)-x_o}{x_\infty - x_o} = 1.0 - \exp(-t/\tau)$



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- Time Constant, τ - time required to complete 63.2% of the process.
- Rise Time, t_r - time required to achieve response from 10% to 90% of final value.
  ↬ For first order system, $t_r = 2.31\tau - 0.11\tau = 2.2\tau$.
- Settling Time, t_s - the time for the response to reach, and stay within 2% of its final value.
  ↬ For first order system, $t_s = 4\tau$.
- Process is assumed to be completed when $t \geq 5\tau$.
- Faster response is associated with shorter τ.

Response of a 1^{st} Order System: Harmonic Input

If the governing equation for first-order system is solved for harmonic input and $x|_{t=0} = 0$, the solution is:

$$\frac{x(t)}{A\mathbb{k}} = \underbrace{\frac{\omega\tau}{1+(\omega\tau)^2}\exp(-t/\tau)}_{\text{transient response}} + \underbrace{\frac{1}{\sqrt{1+(\omega\tau)^2}}\sin(\omega t + \phi)}_{\text{steady-state response}}$$

where, $\phi \equiv \tan^{-1}(-\omega\tau) \equiv$ *phase lag*. Hence, *time delay*, Δt, is related to phase lag as:

$$\Delta t = \frac{\phi}{\omega}$$

For $\omega\tau >> 1$, response is attenuated and time/phase is lagged from input, and for $\omega\tau << 1$, the transient effect becomes very small and response follows the input with small attenuation and time/phase lag.

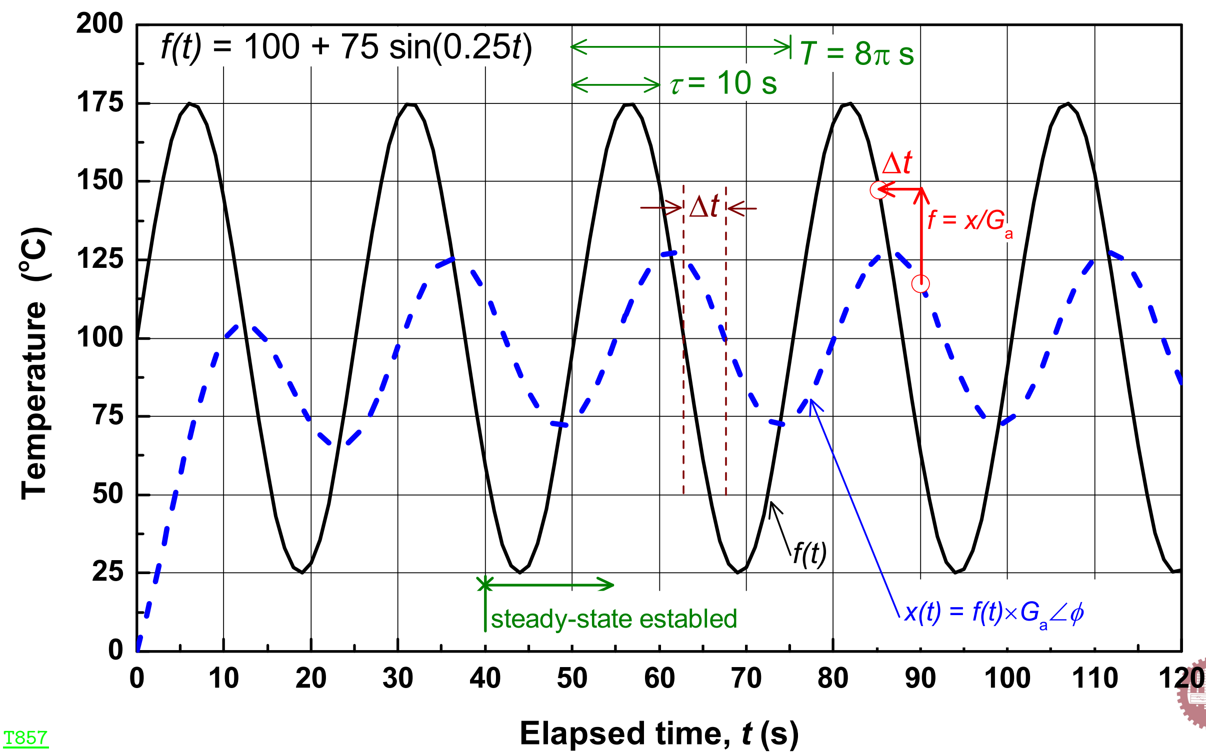
- Ideal response (without attenuation and phase lag) is obtained when the system time constant, τ is significantly smaller than the forcing element period, $T \equiv 2\pi/\omega$.
- As $t \to \infty$, the steady-state solution:

$$x(t)|_s = \frac{A\mathbb{k}}{\sqrt{1+(\omega\tau)^2}}\sin(\omega t + \phi) = G_a f(t)\angle\phi$$

Hence, $G_a \equiv \mathbb{k}/\sqrt{1+(\omega\tau)^2} \equiv$ *steady-state gain.*

- The attenuated steady-state response is also a sine wave with a frequency equal to the input signal frequency, ω, and it lags behind the input by phase angle, ϕ.

Thermometer ($\tau = 10s$), initially at 0°C ($\omega = 0.25$, $T = 8\pi$, $G_a = 0.37$).



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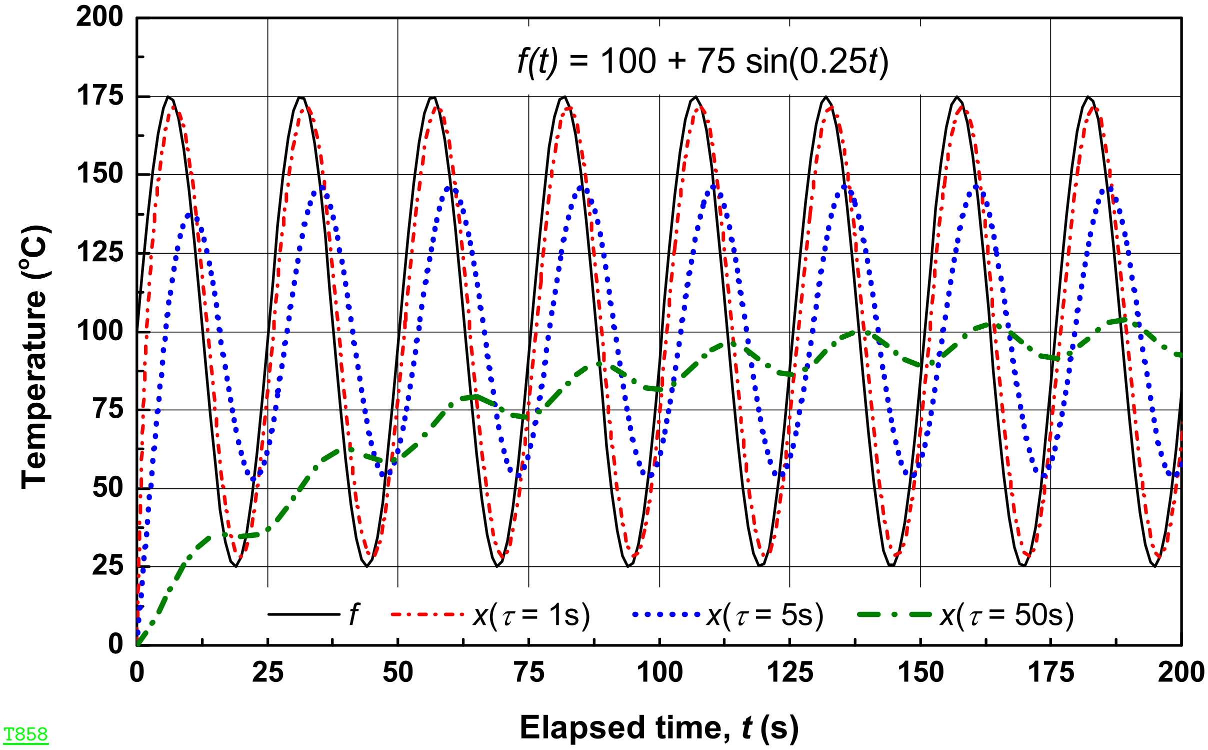

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Effects of time constant on system response.

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Unit	τ [s]	τ/T	ϕ [deg]	Δt [s]	G_a
01	01	0.04	-14.0	-0.98	0.97
02	05	0.2	-51.3	-3.58	0.62
03	50	2.0	-85.4	-5.96	0.08

- Response to harmonic input is
  - at same frequency,
  - with a phase shift (time lag), and
  - reduced amplitude (attenuation).
- The larger the time constant, the greater the time lag & amplitude decrease (attenuation).

TF of a 1^{st} Order System

$$\tau \frac{dx}{dt} + x = \Bbbk\, f(t)$$

- $\frac{d^n x}{dt^n} \Longrightarrow s^n X(s), \qquad f(t) \Longrightarrow F(s).$

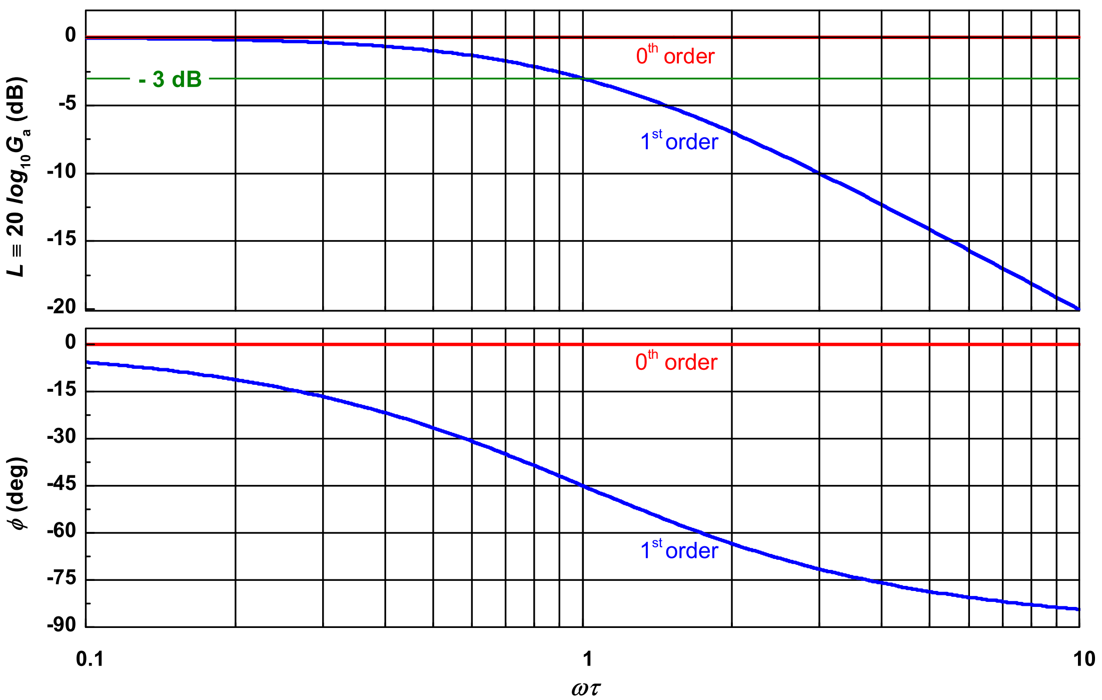
$\Rightarrow \tau s X(s) + X(s) = \Bbbk\, F(s)$

$\Rightarrow \frac{X(s)}{F(s)} = \frac{\Bbbk}{\tau s + 1}$

$$F(s) \Longrightarrow \boxed{\frac{\Bbbk}{\tau s+1}} \Longrightarrow X(s)$$

- $s \Longleftarrow j\omega$
- $G_a = |G(j\omega)| = \left|\frac{\Bbbk}{j\omega\tau+1}\right| = \frac{\Bbbk}{\sqrt{1+(\omega\tau)^2}}$
- $\phi = \angle G(j\omega) = \tan^{-1}(-\omega\tau)$

Bode Diagram of 0^{th} & 1^{st} Order Systems



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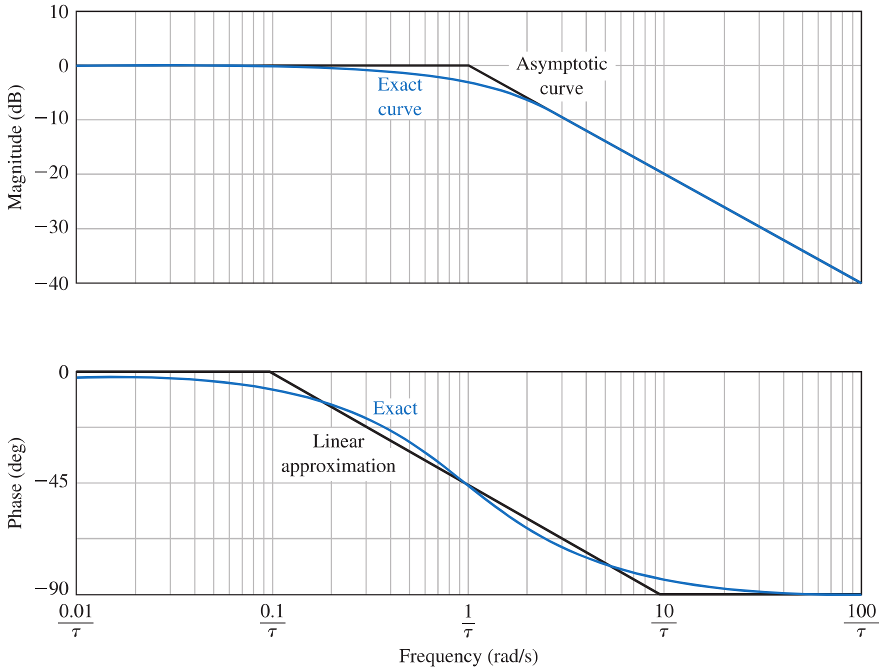
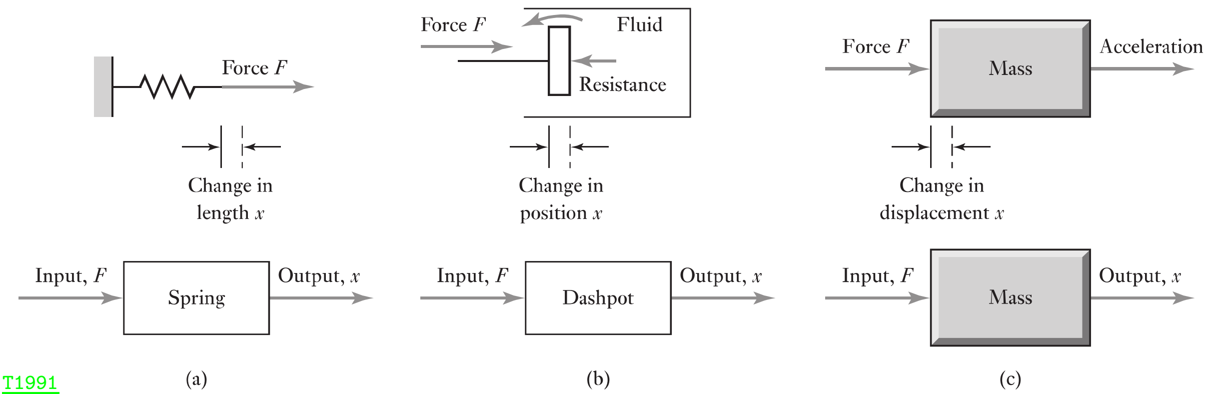


T2006

$$20\log|G_a| = \begin{cases} 0 & \text{for } \omega\tau << 1, \\ 3.01 \text{ dB} & \text{for } \omega\tau = 1, \\ -20\log(\omega\tau) & \text{for } \omega\tau >> 1. \end{cases}$$

Mechanical System Elements



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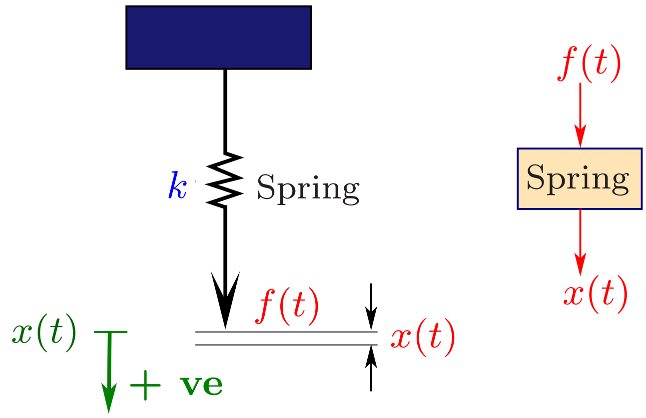
Mechanical systems: (a) spring, (b) dash-pot, (c) mass.

(a) Spring

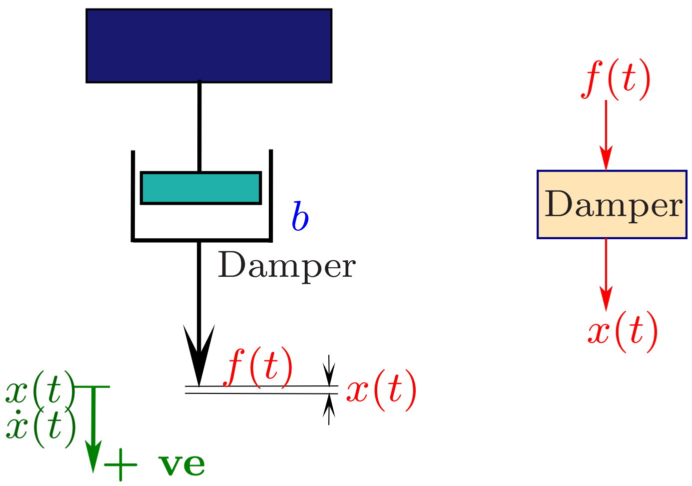


T848

$$\boxed{f(t) = k\ x(t)}$$

F	$\equiv$	Force (tension or compression),
x	$\equiv$	Displacement (extension or compression),
k	$\equiv$	Spring constant. The bigger the value of k the greater the forces required to stretch or compress the spring and so the greater the stiffness.

(b) Dash-pot/Damper

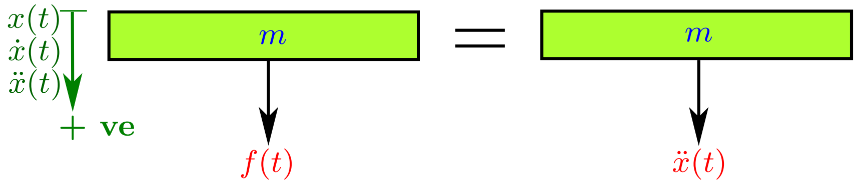


T849

$$\boxed{f(t) = -bv = -b\frac{dx}{dt}}$$

F	$\equiv$	Force opposing the motion at velocity v,
b	$\equiv$	Damping coefficient. Larger the value of b the greater the damping force at a particular velocity.

(c) Mass



T850

$$\boxed{f(t) = ma = m\frac{dv}{dt} = m\frac{d^2x}{dt^2}}$$

$F \equiv$ Force required to cause acceleration, a,

$m \equiv$ Mass of the element that is distributed throughout some volume. Often, it is assumed to be concentrated at a point.

- Spring stores energy when stretched, and the energy is released when it springs back to its original state.

$$E = \frac{1}{2}\frac{f^2}{k}$$

- Energy is stored in mass when it is moving with a velocity, v, the energy being referred to as kinetic energy.

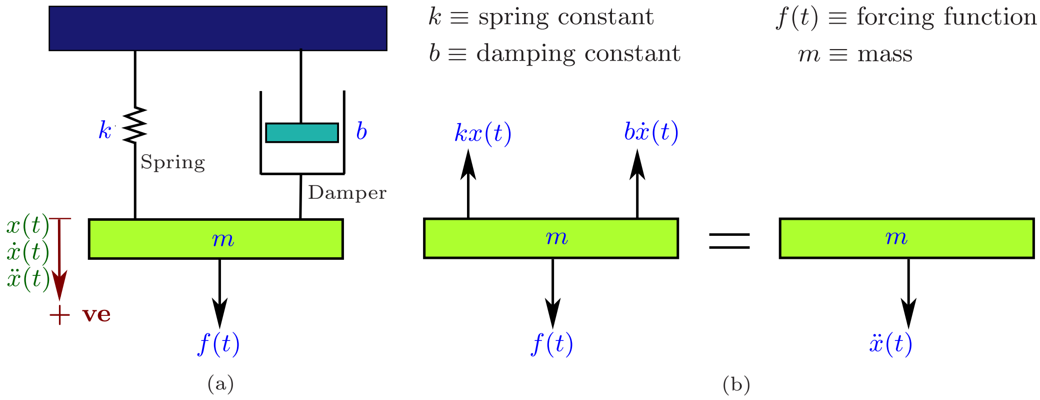
$$E = \frac{1}{2}mv^2$$

- Dashpot dissipates energy as heat rather than storing it, and dissipated power, P depends on the velocity, v.

$$P = bv^2$$

Second Order System (spring-damper-mass)

$k \equiv$ spring constant
$b \equiv$ damping constant
$f(t) \equiv$ forcing function
$m \equiv$ mass



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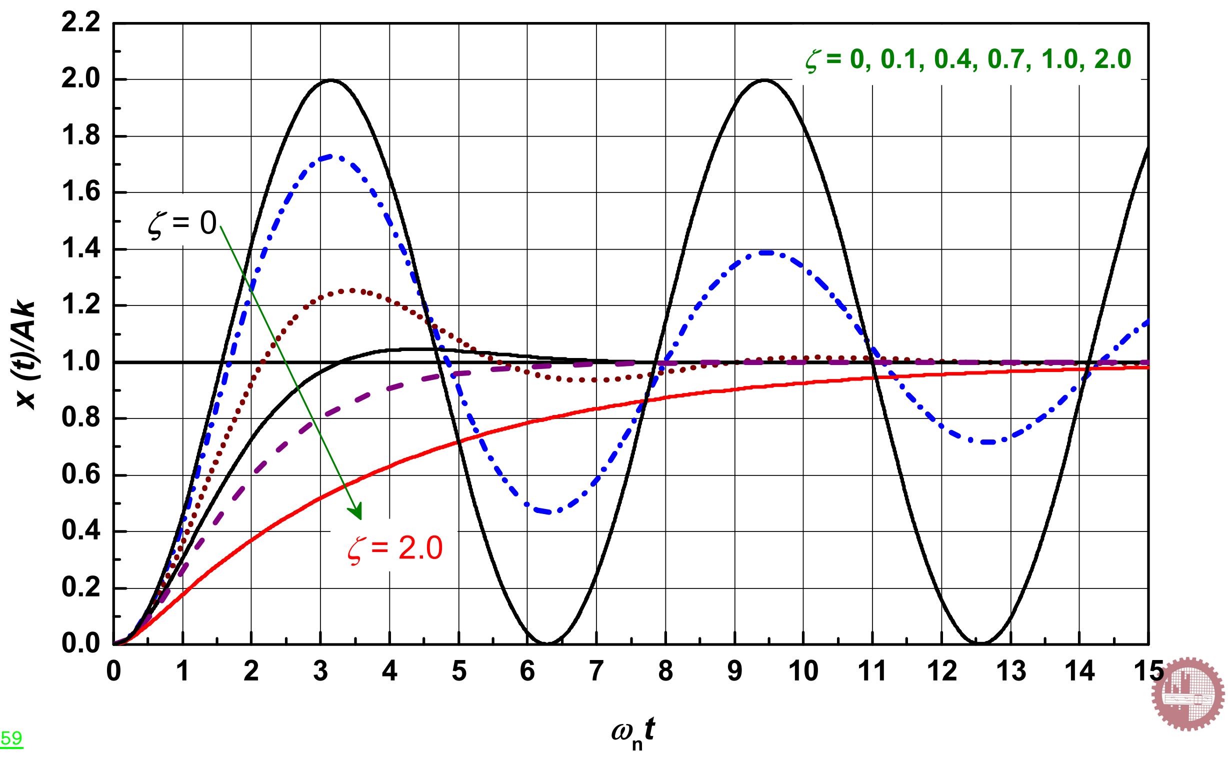
$$f - k\,x - b\frac{dx}{dt} = m\frac{d^2x}{dt^2} \Longrightarrow m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + k\,x = f$$

$$\begin{array}{lcll} \omega_n & \equiv & \sqrt{\frac{k}{m}} & \Longleftrightarrow \text{ undamped natural frequency (rad/s),} \\ b_c & \equiv & 2\sqrt{mk} & \Longleftrightarrow \text{ critical damping coefficient,} \\ \zeta & \equiv & b/b_c & \Longleftrightarrow \text{ damping ratio.} \end{array}$$

Response of a 2^{nd} Order System: Step Input



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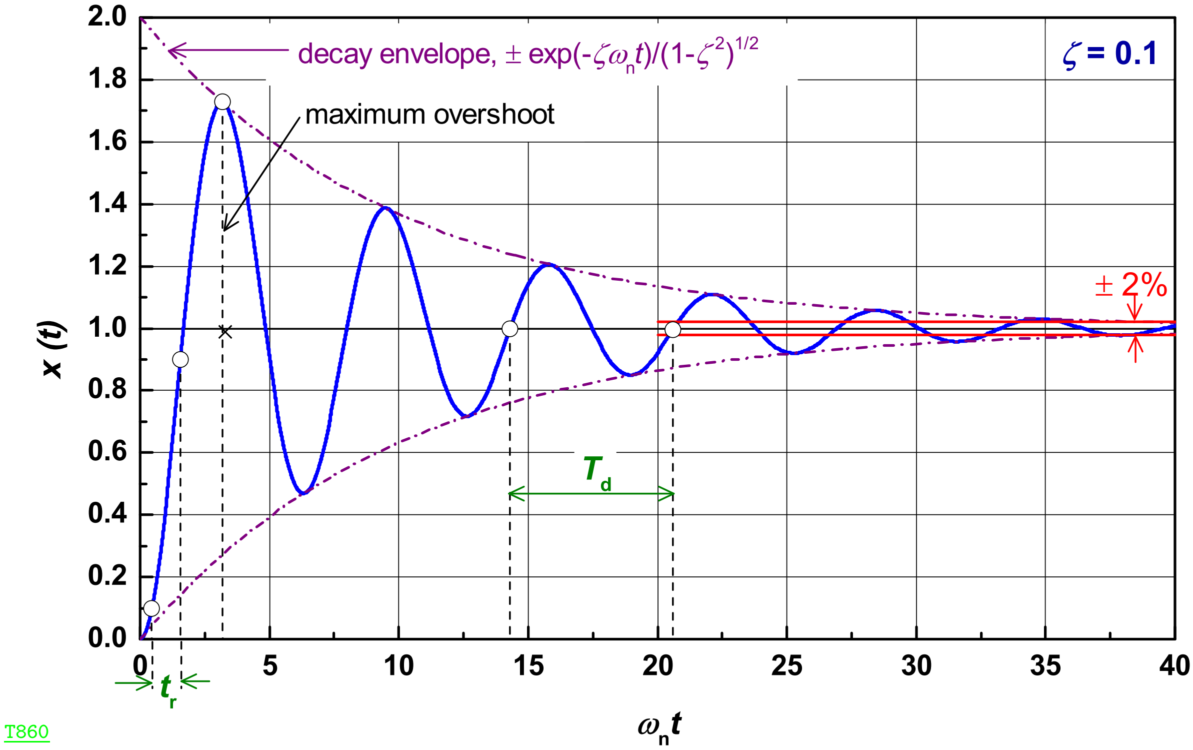


Second-order under-damped response specifications.

- Steady state position is obtained after a long period of time.
- Under-damped system ($\zeta < 1$): response overshoots the steady-state value initially, & then eventually decays to the steady-state value. The smaller the value of ζ, the larger the overshoot. The transient response oscillates about the steady-value and occurs with a period, T_d, given by:

$$T_d \equiv \frac{2\pi}{\omega_d} \quad : \quad \omega_d \equiv \omega_n\sqrt{1-\zeta^2}$$

- Critical damping ($\zeta = 1$): an exponential rise occurs to approach the steady-state value without any overshoot.
- Over-damped ($\zeta > 1$): the system approaches the steady-state value without overshoot, but at a slower rate.

TF of a 2^{nd} Order System

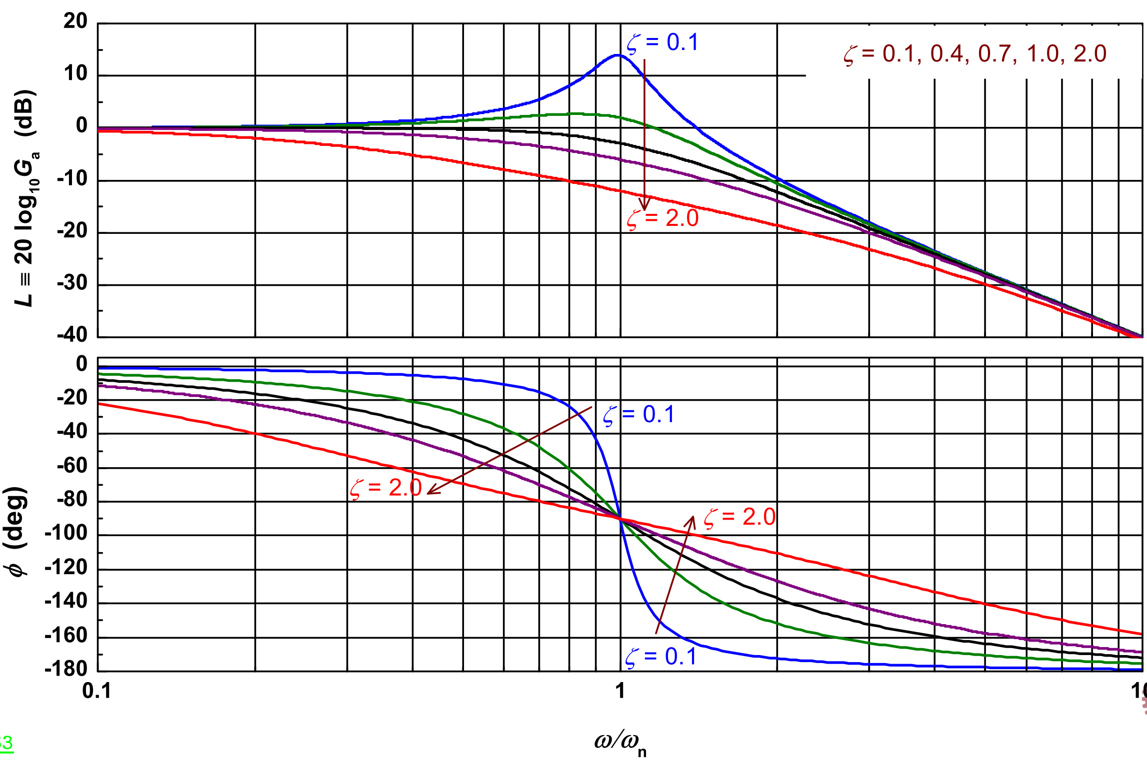
$$\boxed{a_2\frac{d^2x}{dt^2}+a_1\frac{dx}{dt}+a_o x = f(t) :\Longrightarrow \frac{1}{\omega_n^2}\frac{d^2x}{dt^2}+2\frac{\zeta}{\omega_n}\frac{dx}{dt}+x=\mathbb{k}\,f(t)}$$

$$\text{where,}\quad \begin{array}{lll} \mathbb{k} & \equiv 1/a_o & \equiv \text{static sensitivity,} \\ \omega_n & \equiv \sqrt{\frac{a_o}{a_2}} & \equiv \text{undamped natural frequency,} \\ \zeta & \equiv \frac{a_1}{2\sqrt{a_o a_2}} & \equiv \text{dimensionless damping ratio.} \end{array}$$

$$\boxed{G(s)=\frac{1/\mathbb{k}}{\frac{1}{\omega_n^2}s^2+2\frac{\zeta}{\omega_n}s+1}=\frac{\omega_n^2/\mathbb{k}}{s^2+2\zeta\omega_n s+\omega_n^2}}$$

- $G(j\omega)=\dfrac{1/\mathbb{k}}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]+j\left[2\zeta\frac{\omega}{\omega_n}\right]}$
- $G_a=|G(j\omega)|=\dfrac{1/\mathbb{k}}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2}}$
- $\phi=\angle G(j\omega)=\tan^{-1}\left[-\dfrac{2\zeta\frac{\omega}{\omega_n}}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]}\right]$

Response of a 2^{nd} Order System: Harmonic Input



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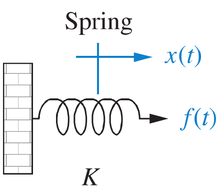
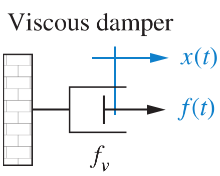
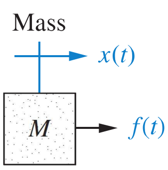
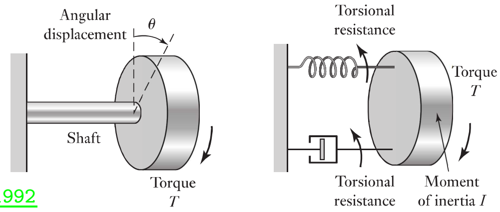
- System has a good *linearity* for low damping ratios ($0 < \zeta \geq 0.3$) since the amplitude gain is very nearly unity ($G_a \simeq 1$).
- For large values of ζ, the amplitude is reduced substantially.
- The phase shift characteristics are a strong function of frequency ratio (ω/ω_n) for all frequencies.
- As a general rule of thumb, the choice of $\zeta = 0.707$ is optimal since it results in the best combination of amplitude linearity and phase linearity over the widest range of frequencies.

Component	Force-velocity	Force-displacement	Impedence $Z_M(s) = F(s)/X(s)$
	$f(t) = K \int_0^t v(\tau) d\tau$	$f(t) = Kx(t)$	K
	$f(t) = f_v v(t)$	$f(t) = f_v \frac{dx(t)}{dt}$	$f_v s$
	$f(t) = M \frac{dv(t)}{dt}$	$f(t) = M \frac{d^2x(t)}{dt^2}$	Ms^2

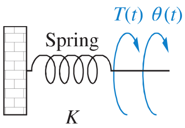
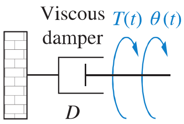
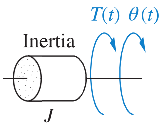
T1986

Modelling of Rotational Mechanical System



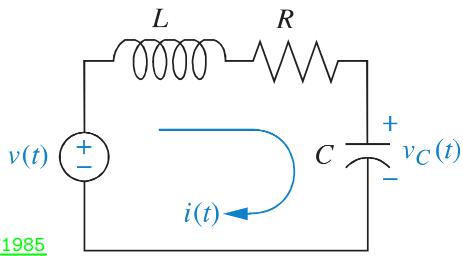
T1992

$$J\frac{d^2\theta(t)}{dt^2} + D\frac{d\theta(t)}{dt} + K\theta(t) = T(t)$$

Component	Torque-angular velocity	Torque-angular displacement	Impedence $Z_M(s) = T(s)/\theta(s)$
	$T(t) = K\int_0^t \omega(\tau)d\tau$	$T(t) = K\theta(t)$	K
	$T(t) = D\omega(t)$	$T(t) = D\frac{d\theta(t)}{dt}$	Ds
	$T(t) = J\frac{d\omega(t)}{dt}$	$T(t) = J\frac{d^2\theta(t)}{dt^2}$	Js^2

T1987

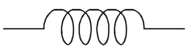
Modelling of Electrical System (R-L-C system)



T1985

- $L\frac{d^2q(t)}{dt^2} + R\frac{dq(t)}{dt} + \frac{1}{C}q(t) = v(t)$
- $L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_o^t i(t)dt = v(t)$
- For capacitor, $q(t) = Cv_c(t)$

$$\Longrightarrow LC\frac{d^2v_c(t)}{dt^2} + RC\frac{dv_c(t)}{dt} + v_c(t) = v(t)$$

Component	Voltage-current	Current-voltage	Voltage-charge	Impedance $Z(s) = V(s)/I(s)$
 Capacitor	$v(t) = \frac{1}{C}\int_0^1 i(\tau)d\tau$	$i(t) = C\frac{dv(t)}{dt}$	$v(t) = \frac{1}{C}q(t)$	$\frac{1}{Cs}$
 Resistor	$v(t) = Ri(t)$	$i(t) = \frac{1}{R}v(t)$	$v(t) = R\frac{dq(t)}{dt}$	R
 Inductor	$v(t) = L\frac{di(t)}{dt}$	$i(t) = \frac{1}{L}\int_0^1 v(\tau)d\tau$	$v(t) = L\frac{d^2q(t)}{dt^2}$	Ls

T1984

- $LC\frac{d^2v_c(t)}{dt^2} + RC\frac{dv_c(t)}{dt} + v_c(t) = v(t)$

$\rightarrow$ $\left[LC\,s^2 + RC\,s + 1\right]\,V_c(s) = V(s)$

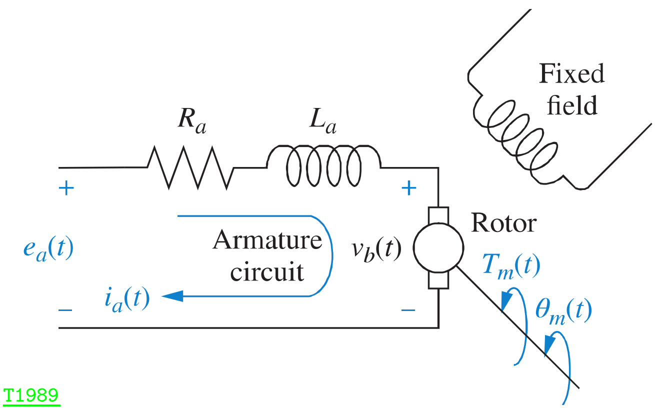
$\Rightarrow$ $\frac{V_c(s)}{V(s)} = \frac{1/LC}{s^2+\frac{R}{L}s+\frac{1}{LC}}$

- $L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_o^t i(t)dt = v(t)$

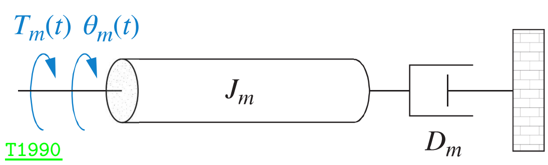
$\rightarrow$ $\left[L\,s + R + \frac{1}{C}\frac{1}{s}\right]\,I(s) = V(s)$

$\Rightarrow$ $\frac{V(s)}{I(s)} = L\,s + R + \frac{1}{Cs}$

Modelling of DC Motors



T1989

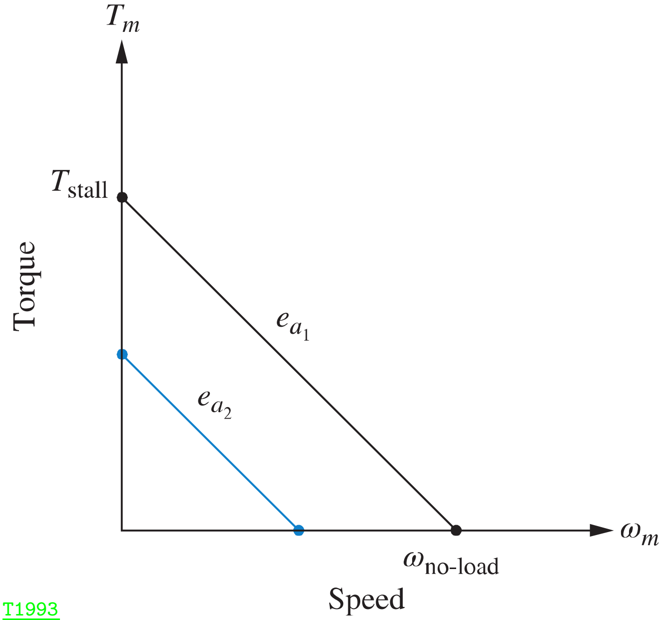


T1990

1. $e_a(t) = R_a\,i_a(t) + L_a\frac{di_a(t)}{dt} + v_b(t)$
2. $v_b(t) = K_b\frac{d\theta_m(t)}{dt}$
3. $T_m(t) = K_t\,i_a(t)$
4. $T_m(t) = J_m\frac{d^2\theta_m(t)}{dt^2} + D_m\frac{d\theta_m(t)}{dt}$

$\otimes$ Eliminate $i_a(t)$.

- For dc motors, the effects of L_a are less than R_a, so $L_a \to 0$.
- $e_a(t) = K_b \frac{d\theta_m(t)}{dt} + \frac{R_a}{K_t} T_m(t) = K_b\, \omega_m(t) + \frac{R_a}{K_t} T_m(t)$
- For dc: motor will turn at constant velocity, ω_m and torque, T_m.
- $\boxed{T_m(t) = -\frac{K_b K_t}{R_a}\omega_m(t) + \frac{K_t}{R_a} e_a(t)} \Rightarrow \boxed{T_m = -\frac{K_b K_t}{R_a}\omega_m + \frac{K_t}{R_a} e_a}$



- Stall torque ($\omega_m = 0$): $\to T_{stall} = \frac{K_t}{R_a} e_a$
- No-load speed ($T_m = 0$): $\to \omega_{no-load} = \frac{e_a}{K_b}$

$\Longrightarrow \boxed{\frac{K_t}{R_a} = \frac{T_{stall}}{e_a}}$

$\Longrightarrow \boxed{K_b = \frac{e_a}{\omega_{no-load}}}$

T1993

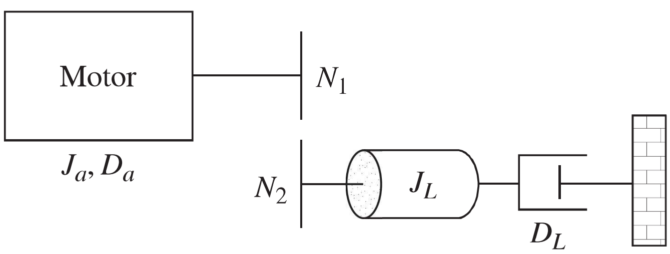
- $T_m(t) = J_m \frac{d^2\theta_m(t)}{dt^2} + D \frac{d\theta_m(t)}{dt} \to T_m(s) = J_m s^2\, \theta_m(s) + D_m s\, \theta(s)$
- $e_a(t) = K_b \frac{d\theta_m(t)}{dt} + \frac{R_a}{K_t} T_m(t) \to E_a(s) = K_b s\, \theta(s) + \frac{R_a}{K_t} T_m(s)$

$\Rightarrow \boxed{\frac{\theta_m(s)}{E_a(s)} = \frac{K_t/R_a J_m}{s^2 + \frac{s}{J_m}(D_m + K_b K_t/R_a)} = \frac{K}{s(s+\alpha)}}$

- $K = \frac{K_t}{R_a J_m}$: $\alpha = \frac{D_m + K_b K_t/R_a}{J_m}$

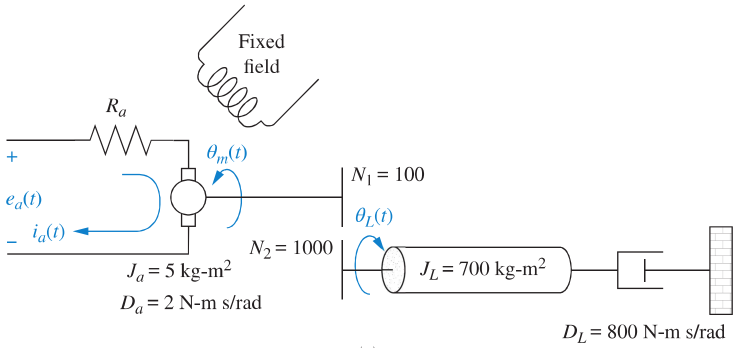
DC motor driving a rotational mechanical load

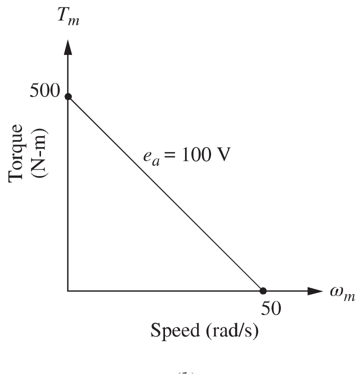
- $J_m = J_a + J_L \left(\frac{N_1}{N_2}\right)^2$: $D_m = D_a + D_L \left(\frac{N_1}{N_2}\right)^2$



T2007

Example: ▷ Find a) $\theta_m(s)/E_a(s)$ b) $\theta_l(s)/E_a(s)$.





T2008

- $T_{stall} = 500,\ \omega_{no-load} = 50,\ e_a = 100.$

$\Rightarrow \frac{K_t}{R_a} = \frac{T_{stall}}{e_a} = 5,\ K_b = \frac{e_a}{\omega_{no-load}} = 2.$

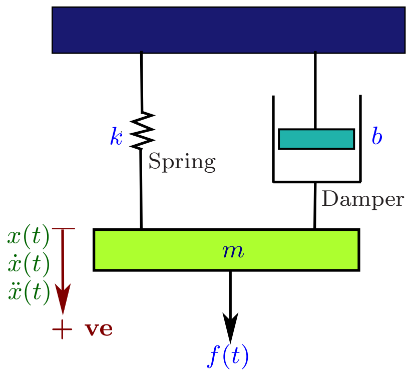
- $J_m = J_a + J_L\left(\frac{N_1}{N_2}\right)^2 = 5 + 700(1/10)^2 = 12,\ D_m = D_a + D_L\left(\frac{N_1}{N_2}\right)^2 = 10.$
- $K = \frac{K_t}{R_a J_m} = 5/12 = 0.417;\ \alpha = \frac{D_m + K_b K_t/R_a}{J_m} = \frac{10+5(2)}{12} = 1.667.$

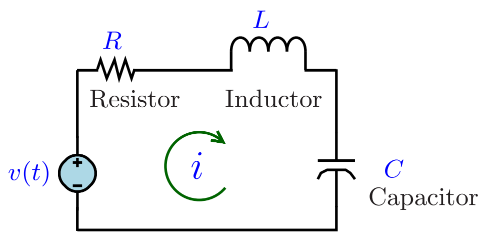
$\Rightarrow \frac{\theta_m(s)}{E_a(s)} = \frac{K}{s(s+\alpha)} = \frac{0.417}{s(s+1.667)};\ \frac{\theta_l(s)}{E_a(s)} = \frac{0.0417}{s(s+1.667)},$ as $\theta_L(t) = \theta_m(t)(N_m/N_L)$

Spring-mass-damper system & analogous RLC circuit

$f(t) \equiv$ forcing function (N)
$m \equiv$ mass (kg)
$k \equiv$ spring constant (N/m)
$b \equiv$ damping constant (N.s/m)
$x \equiv$ displacement (m)
$\dot{x} \equiv dx/dt \equiv$ velocity (m/s)

$v(t) \equiv$ applied voltage (V)
$L \equiv$ inductance (H)
$C \equiv$ capacitance (F)
$R \equiv$ resistance (Ω)
$q \equiv$ charge (C)
$i \equiv dq/dt \equiv$ current (A)





$$\boxed{L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = v(t)}$$

$L \sim m, R \sim b, \frac{1}{C} \sim k, v \sim f$

$$\boxed{m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = f(t)} \Longrightarrow \boxed{\frac{1}{\omega_n^2}\frac{d^2x}{dt^2} + 2\frac{\zeta}{\omega_n}\frac{dx}{dt} + x = \kappa f(t)}$$

T852

General Features

- Most system components fit into one of following five types:
  1. electrical,
  2. mechanical,
  3. liquid flow,
  4. gas flow, or
  5. thermal.
- The behaviour of components of one type is analogous to behaviour of components of any other type. Its is determined by four elements that are common to 5 types of components:
  1. resistance,
  2. capacitance,
  3. inertia (or inductance or), and
  4. dead-time delay.

⊙ For each type of component, the four elements are defined in terms of three variables:

1. The first variable defines a quantity of material, energy, or distance.
2. The second variable defines a driving force or potential that tends to move or change the quantity variable.
3. The third variable is time.

	Variable		
Type of Component	Quantity	Potential	Time
Electrical	Charge	Voltage	Second
Liquid flow	Volume	Pressure	Second
Gas flow	Mass	Pressure	Second
Thermal	Heat energy	Temperature	Second
Mechanical	Distance	Force	Second

Resistance

- Resistance is an opposition to the movement or flow of material or energy. It is measured in terms of the amount of potential required to produce one unit of electric current, liquid flow rate, gas flow rate, heat flow rate, or velocity.
- Electrical resistance is the increase in the voltage across the terminals of a component required to move one more coulomb/second of charge through the component.
- Liquid flow resistance is the increase in pressure drop between two points along a pipe required to increase 1 m^3/s flow rate through the pipe.
- Gas flow resistance is the increase in pressure drop between two points along a pipe required to increase 1 kg/s flow rate through the pipe.
- Thermal resistance is the increase in temperature difference across a wall section required to increase 1 J/s heat flow through the wall section.
- Mechanical resistance is the change in the force applied to an object required to increase 1 m/s velocity of the object.

Capacitance

- Capacitance is measured in terms of the amount of material, energy, or distance required to make a unit change in potential.
- Electrical capacitance is the coulombs of charge that must be stored in a capacitor to increase its voltage by 1 Volt.
- Liquid capacitance is the cubic meters of liquid that must be added to a tank to increase the pressure by 1 Pascal.
- Gas capacitance is the kilograms of gas that must be added to a tank to increase the pressure by 1 Pascal.
- Thermal capacitance is the amount of thermal energy that must be added to an object to increase its temperature by 1°C.
- Mechanical capacitance is the amount of compression of a spring (in meters) required to increase the spring force by 1 Newton.

Inertia, inertance, or inductance

- Inertia, inertance, or inductance is an opposition to a change in the state of motion. It is measured in terms of the amount of potential required to increase electric current, liquid flow rate, gas flow rate, or velocity by one unit per second.
- Electrical inductance is the increase in voltage across an inductor required to increase the current by 1 ampere/s.
- Liquid flow inertance is the increase in the pressure drop between two points along a pipe required to accelerate the flow-rate by 1 m^3/s/s.
- Mechanical inertia is the increase in force required to produce an acceleration of 1 m/s^2.

Dead time (t_d)

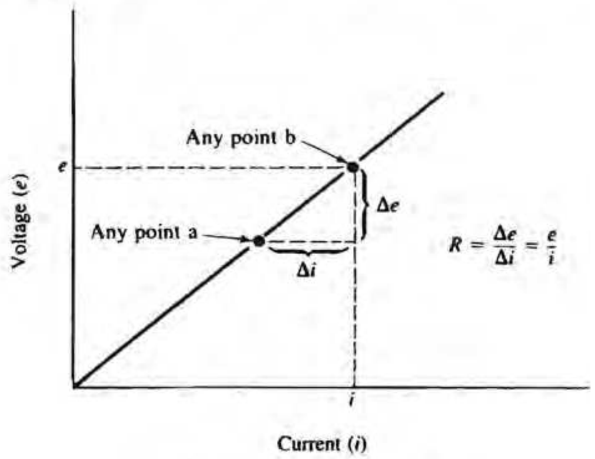
- Dead time is the time interval between the time a signal appears at the input of a component and the time the corresponding response appears at the output.
- Dead time occurs whenever mass or energy is transported from one point to another. It is the time required for the mass or energy to travel from the input location to the output location.
- If v is the velocity of the mass or energy and D is the distance travelled, dead-time delay (t_d) is given by:

$$t_d = \frac{D}{v}$$

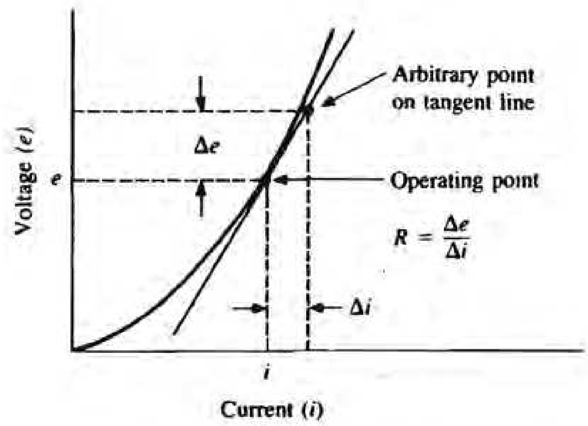
- In general: $f_o(t) = f_i(t - t_d)$

Electrical Elements

- Electrical resistance is that property of material which impedes the flow of electric current. The unit of electric resistance is the ohm.



T1997 a) Resistance of a linear element



T1998 b) Resistance of a nonlinear element

Static resistance, $R = \frac{e}{i}$

Dynamic resistance, $R = \frac{\Delta e}{\Delta i} = \frac{de}{di}$

- Electrical capacitance is the quantity of electric charge (C) required to make a unit increase in the electrical potential (eV). The unit of electrical capacitance is the farad (F). Capacitance, $C = \frac{\Delta q}{\Delta e}$.

$\Rightarrow \Delta q = C\Delta e \rightarrow \frac{\Delta q}{\Delta t} = i = C\frac{\Delta e}{\Delta t} \rightarrow \frac{dq}{dt} = i = C\frac{de}{dt}$

$\rightarrow$ If $e = A\sin\omega t \rightarrow \frac{de}{dt} = \omega A\cos\omega t \rightarrow i = \omega CA\cos\omega t$

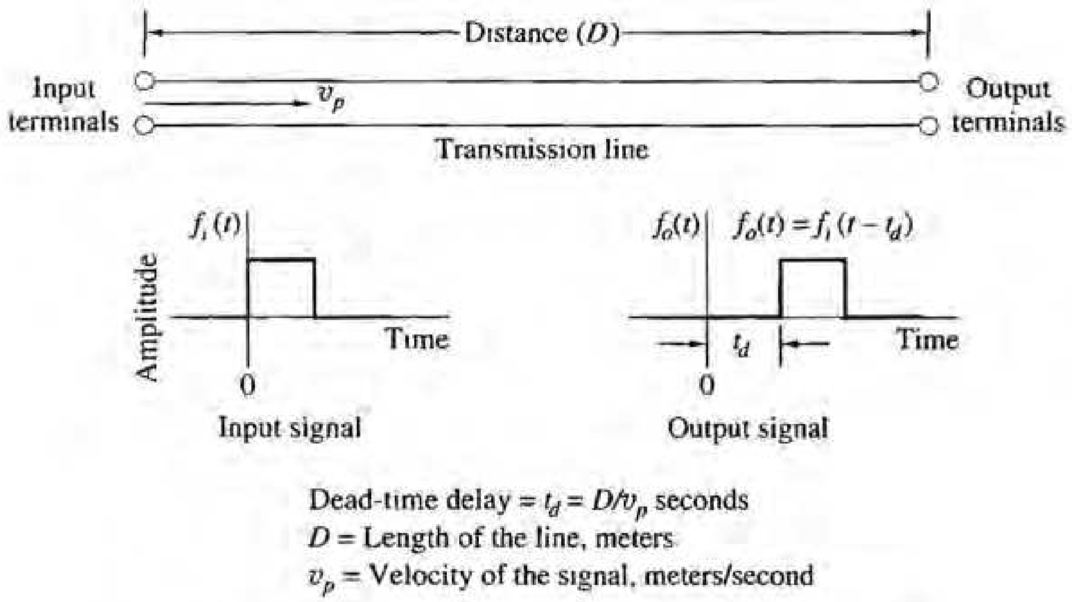
- Electrical inductance is the voltage required to produce a unit increase in electric current each second. The unit of electrical inductance is the henry (H).

$\Rightarrow e = L\frac{\Delta i}{\Delta t} = L\frac{di}{dt}$

$\rightarrow$ If $i = A\sin\omega t \rightarrow \frac{di}{dt} = \omega A\cos\omega t \rightarrow e = \omega LA\cos\omega t$

- Electrical dead-time delay is the delay caused by the time it takes a signal to travel from the source to the destination. Dead-time delay of the line is equal to the distance the signal travels (D) divided by the velocity of propagation (v_p).

$\Rightarrow t_d = \frac{D}{v_p}$

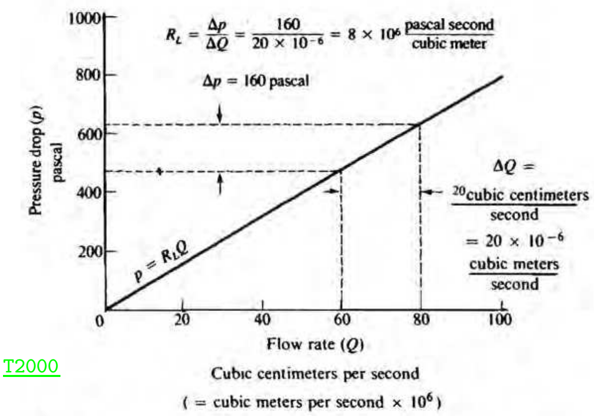


T1999

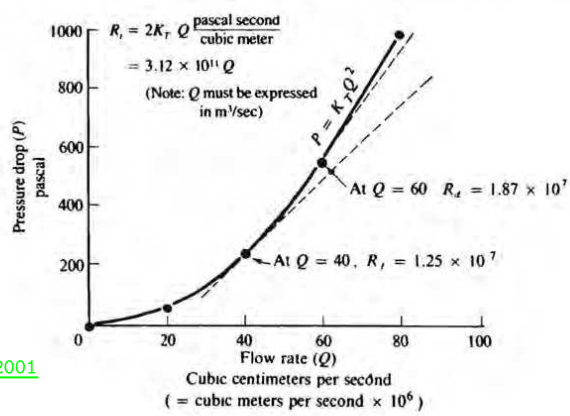
Example: ▷ Determine the dead-time delay of a 600-m-long transmission line if the velocity of propagation is 2.3 X 10^8 m/s. [2.61 μs]

Liquid Flow Elements

- Liquid flow resistance is that property of pipes, restrictions , or valves which impedes the flow of a liquid. It is measured in terms of the increase in pressure required to make a unit increase in flow rate. The SI unit is Pa.s/m.



T2000



T2001

Laminar flow:

$$p = R_L Q : \quad R_L = \frac{128\mu L}{\pi d^4}$$

Turbulent flow:

$$p = K_t Q^2 : \quad R_t = 2K_t Q : \quad K_t = \frac{8\rho f l}{\pi^2 d^5}$$

- Liquid flow capacitance is defined in terms of the increase in volume of liquid in a tank required to make a unit increase in pressure at the outlet of the tank.
- Liquid capacitance, $C_L = \frac{\Delta V}{\Delta p}$.
- $\Delta p = \rho g \Delta H$, and $\Delta H = \frac{\Delta V}{A}$

$$\Rightarrow \boxed{C_L = \frac{A}{\rho g}}$$

Example: ▷ A water tank has a diameter of 1.83 m and a height of 3.28 m. Determine the capacitance of the tank containing water. [2.68×10^{-4} m^3/Pa]

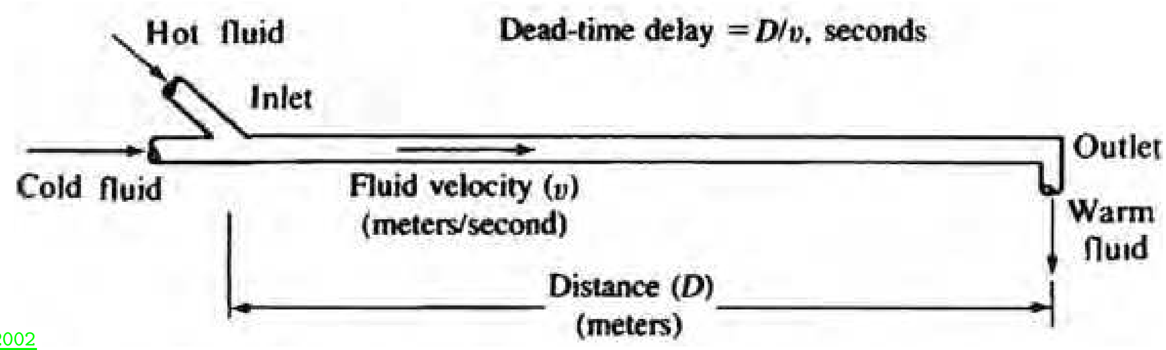
System Elements Liquid Flow Elements

- Liquid flow inertance is measured in terms of the amount of pressure drop in a pipe required to increase the flow rate by 1 unit each second.
- Liquid inertance, $I_L = \frac{p}{\Delta Q / \Delta t}$.
- $m = \rho a l$: $m \equiv$ mass of fluid in pipe, $a \equiv$ x-sectional area of pipe,
- $F = pa = m\frac{\Delta v}{\Delta t}$: $l \equiv$ length of the pipe,
- $\Delta Q = a \Delta v$

$$\Rightarrow \boxed{I_L = \frac{\rho l}{a}}$$

Example: ▷ Determine the liquid flow inertance of water in a pipe that has a diameter of 2.1 cm and a length of 65 m. [1.88E+08 $Pa/m^3/s^2$]

- Dead time occurs whenever liquid is transported from one point to another in a pipeline. The dead-time delay (t_d) is the distance travelled (D) divided by the average velocity (v) of the fluid. $t_d = \frac{D}{v}$: $v = \frac{Q}{A}$.



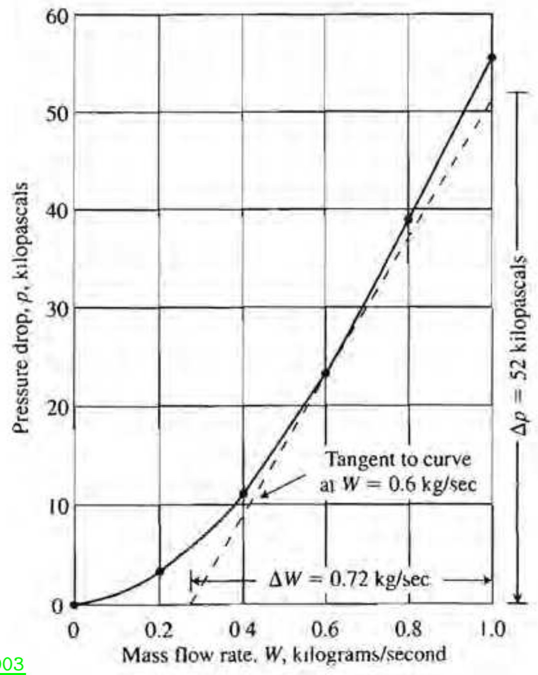
T2002

Example: ▷ Liquid flows in a pipe that is 200 m long and has a diameter of 6 em. The flow rate is 0.0113 m^3/s. Determine the dead-time delay. [50 s]

Gas Flow Elements

- Gas flow resistance is that property of pipes, valves, or restrictions that impedes the flow of a gas. It is measured in terms of the increase in pressure required to produce an increase in 1 kg/s gas flow rate. The SI unit for gas flow resistance is Pa s/kg.
- In practice, gas flow is almost always turbulent, and the commonly used equations apply to turbulent flow. If the pressure drop is less than 10% of the initial gas pressure, the equation for incompressible flow gives reasonable accuracy for gas flow.
  - $p = P_1 - P_2 = K_g W^2$; $W \equiv$ gas flow rate (kg/s).
  - $R_g = 2K_g W$
  - $K_g = \frac{8fl}{\pi^2 d^5 \rho}$

Determine R_g at $\dot{m} = 0.6$ kg/s.



T2003

$$R_g = \frac{p}{\Delta W} = \frac{52000}{0.72} = 72222 \text{Pa s /kg.}$$

- Gas flow capacitance is defined in terms of the increase in the mass of gas in a vessel required to produce a unit increase in pressure while the temperature remains constant. The SI unit of gas flow capacitance is kg/Pa.
- Gas capacitance, $C_g = \frac{\Delta m}{\Delta p}$
- Gas law, $pV = m\frac{R_u}{M}T \rightarrow m = \left(\frac{1.2\times10^{-4}MV}{T}\right)p$

$\Rightarrow$ $C_g = \frac{1.2\times10^{-4}MV}{T}$

Example: ▷ A pressure tank has a volume of 0.75 m^3. Determine the capacitance of the tank if the gas is nitrogen at 20°C. [8.6×10^{-6} kg/Pa]

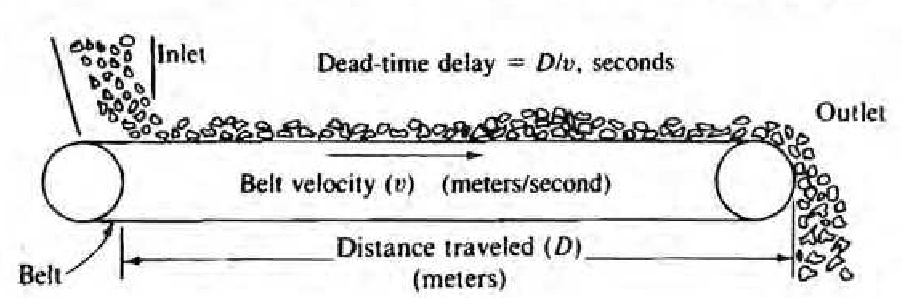
Thermal Elements

- Thermal resistance is that property of a substance that impedes the flow of heat. It is measured in terms of the difference in temperature required to produce a heat flow rate of 1 W.
- Thermal resistance, $R_T = \frac{\Delta T}{Q}$.
- Thermal capacitance is defined in terms of the increase in heat required to make a unit increase in temperature. The SI unit of thermal capacitance is J/K. The thermal capacitance (C_T) of an object is simply the product of the mass (m) of the object times the heat capacity (c) of its substance.
- Thermal capacitance, $C_T = m\ c$.

Mechanical Elements

- Mechanical resistance (or friction) is that property of a mechanical system that impedes motion. It is measured in terms of increase in force required to produce an increase in velocity of 1 m/s. The SI unit of it is Ns/m.
- Mechanical Resistance, $R_m = \frac{F}{v}$
- Mechanical capacitance is defined as the increase in the displacement of a spring required to make a unit increase in spring force. The SI unit of mechanical capacitance is the N/m. The reciprocal of the capacitance is called the spring constant, k.
- Mechanical capacitance, $C_m = \frac{\Delta x}{\delta F} = \frac{1}{k}$
- Mechanical inertia (mass) is measured in terms of the force required to produce a unit increase in acceleration. It is defined by Newton's law of motion, and the term mass is used for the inertia element.
- $F_{av} = m\frac{\Delta v}{\Delta t} \rightarrow F = m\frac{dv}{dt}$

- Mechanical dead time is the time required to transport material from one place to another.



T2005

Example: ▷ A belt conveyor is 30 m long and has a belt speed of 3 m/s. Determine the dead-time delay between the input and output ends of the belt. [10 s]