DRAFT

Feedback Controllers and Final Control Elements

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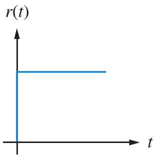
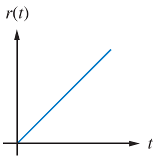
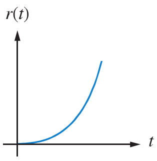
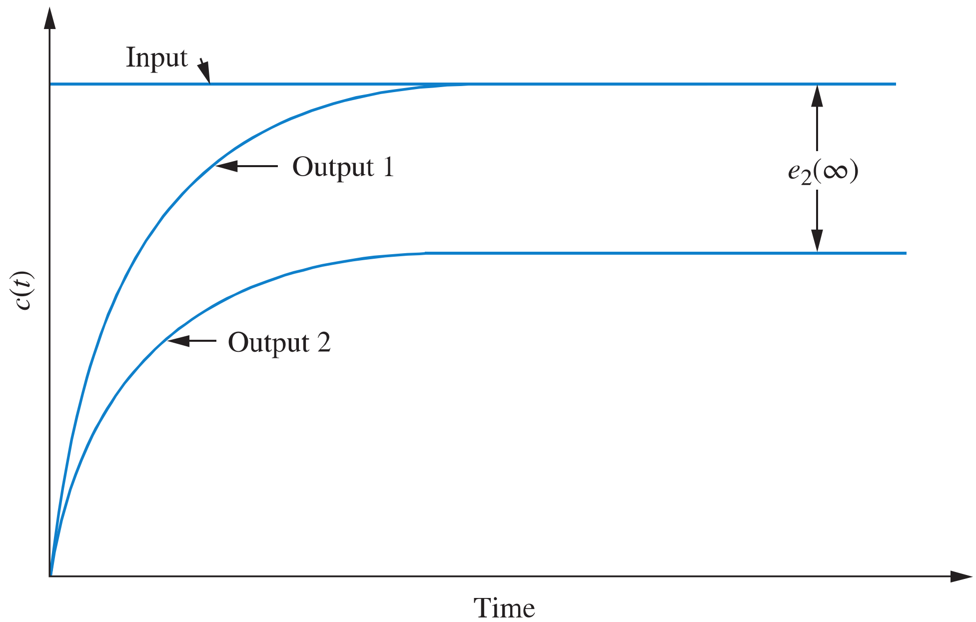
Steady-state Error

Steady-state error (e_{ss}) is the difference between the input and the output for a prescribed test input as $t \to \infty$.

Test waveforms for steady-state errors of position control systems.

Waveform	Name	Physical interpretation	Time function	Laplace transform
	Step	Constant position	1	$\frac{1}{s}$
	Ramp	Constant velocity	t	$\frac{1}{s^2}$
	Parabola	Constant acceleration	$\frac{1}{2}t^2$	$\frac{1}{s^3}$

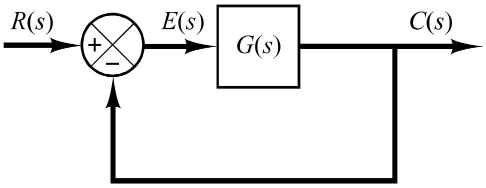
T2043



T2044

Steady-state error for step input

Steady-state Errors in Feedback Control System



T2054

- $e(t) = r(t) - c(t) \quad \rightarrow \quad E(s) = R(s) - C(s)$
- $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} \quad \rightarrow \quad \frac{E(s)}{R(s)} = \frac{1}{1+G(s)}$

$$G(s) = \frac{K(T_a s + 1)(T_b s + 1)\cdots(T_m s + 1)}{s^N (T_1 s + 1)(T_2 s + 1)\cdots(T_p s + 1)}$$

A system is called type 0, type 1, type 2, type p, if N=0, N=1, N=2, N = p, respectively.

- Applying final-value theorem:

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{sR(s)}{1+G(s)}$$

Static Position Error Constant, K_p

- Steady-state error for a unit-step input ($R(s) = 1/s$) is:

$$e_{ss} = \lim_{s\to 0} \frac{s}{1+G(s)} \frac{1}{s} = \frac{1}{1+G(0)}$$

$$K_p \equiv \lim_{s\to 0} G(s) = G(0) \implies e_{ss} = \frac{1}{1+K_p}$$

- Type 0 system:

$$K_p = \lim_{s\to 0} \frac{K(T_a s + 1)(T_b s + 1)\cdots(T_m s + 1)}{(T_1 s + 1)(T_2 s + 1)\cdots(T_p s + 1)} = K$$

- Type 1 or higher system:

$$K_p = \lim_{s\to 0} \frac{K(T_a s + 1)(T_b s + 1)\cdots(T_m s + 1)}{s^N (T_1 s + 1)(T_2 s + 1)\cdots(T_p s + 1)} = \infty$$

$$e_{ss} = \begin{cases} \frac{1}{1+K} & \text{for type 0 systems,} \\ 0 & \text{for } N \geq 1. \end{cases}$$

Static Velocity Error Constant, K_v

- Steady-state error for a unit-ramp input ($R(s) = 1/s^2$) is:

$$e_{ss} = \lim_{s \to 0} \frac{s}{1 + G(s)} \frac{1}{s^2} = \lim_{s \to 0} \frac{1}{sG(s)}$$

$$K_v \equiv \lim_{s \to 0} sG(s) \implies e_{ss} = \frac{1}{K_v}$$

- Type 0 system:

$$K_v = \lim_{s \to 0} \frac{sK(T_a s + 1)(T_b s + 1) \cdots (T_m s + 1)}{(T_1 s + 1)(T_2 s + 1) \cdots (T_p s + 1)} = 0$$

- Type 1 system:

$$K_v = \lim_{s \to 0} \frac{sK(T_a s + 1)(T_b s + 1) \cdots (T_m s + 1)}{s(T_1 s + 1)(T_2 s + 1) \cdots (T_p s + 1)} = K$$

- For a type 2 or higher system:

$$K_v = \lim_{s \to 0} \frac{sK(T_a s + 1)(T_b s + 1) \cdots (T_m s + 1)}{s^N(T_1 s + 1)(T_2 s + 1) \cdots (T_p s + 1)} = \infty$$

$$e_{ss} = \begin{cases} \infty & \text{for type 0 systems,} \\ \frac{1}{K} & \text{for type 1 systems,} \\ 0 & \text{for type 2 or higher systems.} \end{cases}$$

- Type 0 system is incapable of following a ramp input in the steady state.
- Type 1 system with unity feedback can follow the ramp input with a finite error.
- Type 2 or higher system can follow a ramp input with zero error at steady state.

Static Acceleration Error Constant, K_a

- Steady-state error for a unit-parabolic input ($R(s) = 1/s^3$) is:

$$e_{ss} = \lim_{s\to 0} \frac{s}{1+G(s)} \frac{1}{s^3} = \lim_{s\to 0} \frac{1}{s^2 G(s)}$$

$$K_a \equiv \lim_{s\to 0} s^2 G(s) \implies e_{ss} = \frac{1}{K_a}$$

- For a type 0 system:

$$K_a = \lim_{s^2\to 0} \frac{sK(T_a s+1)(T_b s+1)\cdots(T_m s+1)}{(T_1 s+1)(T_2 s+1)\cdots(T_p s+1)} = 0$$

- For a type 1 system:

$$K_a = \lim_{s\to 0} \frac{s^2 K(T_a s+1)(T_b s+1)\cdots(T_m s+1)}{s(T_1 s+1)(T_2 s+1)\cdots(T_p s+1)} = 0$$

- For a type 2 system:

$$K_a = \lim_{s\to 0} \frac{s^2 K(T_a s+1)(T_b s+1)\cdots(T_m s+1)}{s^2(T_1 s+1)(T_2 s+1)\cdots(T_p s+1)} = K$$

- For a type 3 or higher system:

$$K_a = \lim_{s\to 0} \frac{s^2 K(T_a s+1)(T_b s+1)\cdots(T_m s+1)}{s^N(T_1 s+1)(T_2 s+1)\cdots(T_p s+1)} = \infty$$

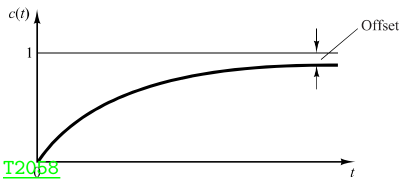
$$e_{ss} = \begin{cases} \infty & \text{for type 0 \& type 1 systems,} \\ \frac{1}{K} & \text{for type 2 systems,} \\ 0 & \text{for type 3 or higher systems.} \end{cases}$$

- Both type 0 and type 1 systems are incapable of following a parabolic input in the steady state.
- Type 2 system with unity feedback can follow a parabolic input with a finite error signal.
- Type 3 or higher system with unity feedback follows a parabolic input with zero error at steady state.

Steady-state error (e_{ss})

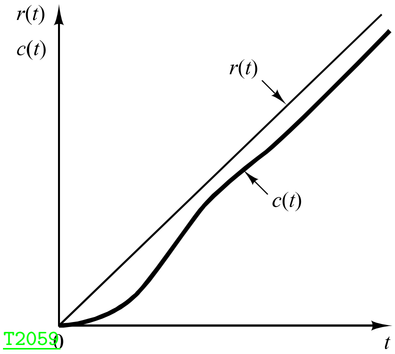
	Step Input $r(t) = 1$	Ramp Input $r(t) = t$	Acceleration Input $r(t) = \frac{1}{2}t^2$
Type 0 system	$\frac{1}{1+K}$	∞	∞
Type 1 system	0	$\frac{1}{K}$	∞
Type 2 system	0	0	$\frac{1}{K}$

T2049



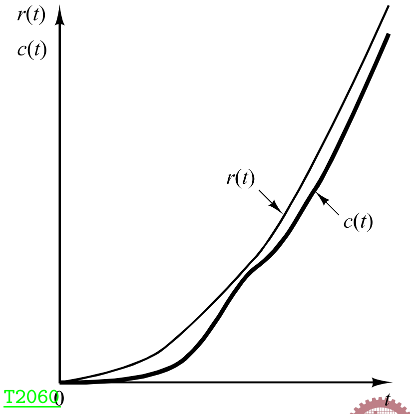
T2058

Type 0 response



T2059

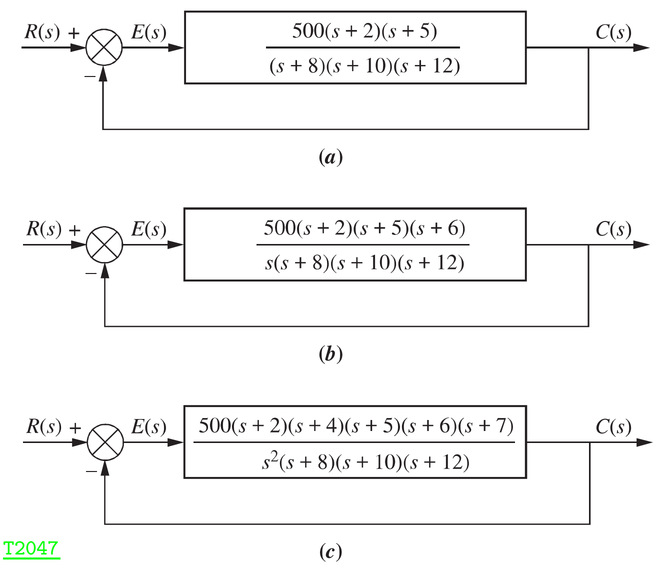
Type 1 response



T2060

Type 2 response

Example: ▷ Steady-State Error via Static Error Constants for standard step, ramp and parabolic input.



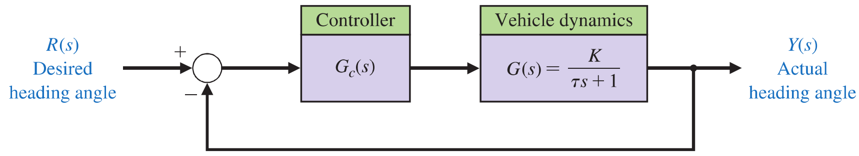
T2047

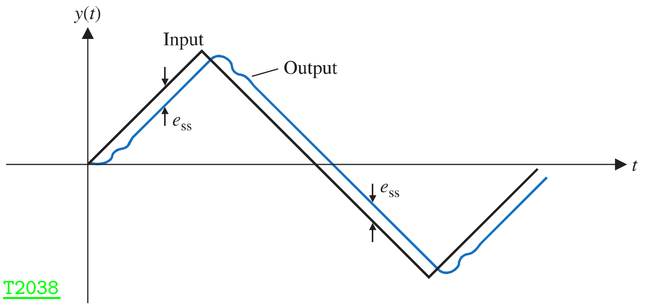
(a):
- step input, $e_{ss} = 0.161$
- ramp input, $e_{ss} = \infty$
- parabolic input, $e_{ss} = \infty$

(b):
- step input, $e_{ss} = 0$
- ramp input, $e_{ss} = 0.032$
- parabolic input, $e_{ss} = \infty$

(c):
- step input, $e_{ss} = 0$
- ramp input, $e_{ss} = 0$
- parabolic input, $e_{ss} = 1.14 \times 10^{-3}$

Example: ▷ Mobile robot steering control, $G_c = K_1 + \frac{K_2}{s}$.



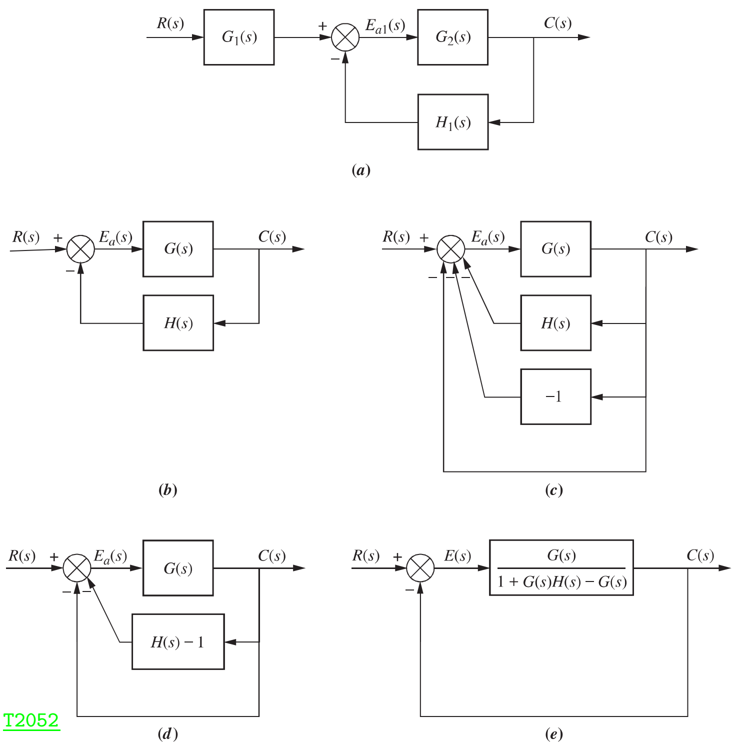


T2038

- $G(s) = \frac{K(K_1 s + K_2)}{s(\tau s + 1)}$
- type 1 system.

- For step input: $e_{ss} = \dfrac{1}{1 + K_p} = 0$; $K_p = \lim_{s \to 0} G(s) = \infty$
- For ramp input: $e_{ss} = \frac{1}{K_v}$; $K_v = \lim_{s \to 0} sG(s) = K_2 K$
- For a triangular wave input: plot above.

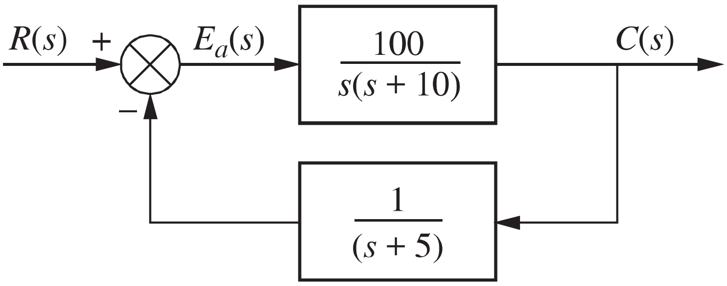
Steady-State Error for Non-unity Feedback Systems



T2052

- Fig.(b): $G(s) = G_1(s)G_2(s)$, and $H(s) = H_1(s)/G_1(s)$.
- Fig.(c): form a unity feedback system by adding and subtracting unity feedback paths.
- Fig.(d): combine H(s) with the negative unity feedback.
- Fig.(e): combine the feedback system consisting of $G(s)$ and $[H(s) - 1]$.

Example: ▷ Steady-State Error for Non-unity Feedback Systems, step input.

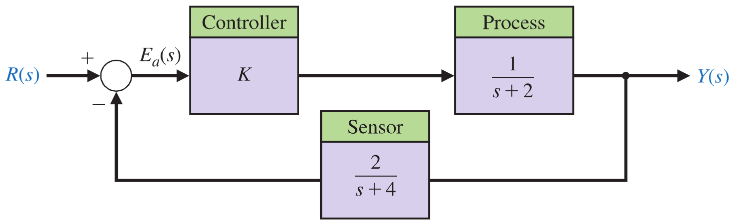


T2053

$$G_e = \frac{G(s)}{1 + G(s)H(s) - G(s)} = \frac{100(s+5)}{s^3 + 15s^2 - 50s - 400}$$

- No pure integrator, type 0.
- $K_p = \lim_{s \to 0} G_e(s) =$ -5/4.
- $e_{ss} = \frac{1}{1+K_p} =$ -4.
- Negative value for steady-state error implies that the output step is larger than the input step.

Example: ▷ Nonunity feedback control system: Estimate K for $e_{ss} = 0$.

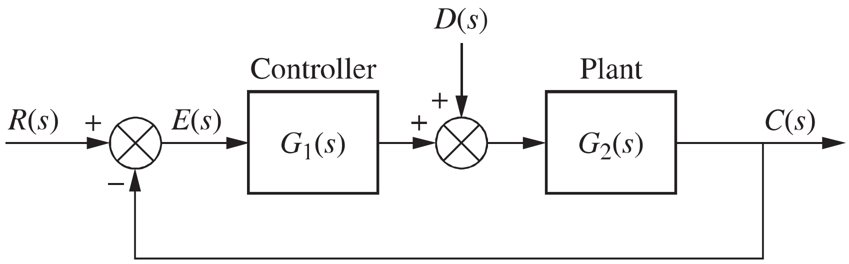


T2039

$$G_e = \frac{G(s)}{1 + G(s)H(s) - G(s)} = \frac{K(s+4)}{s^2 + (6-K)s + (8-2K)}$$

- No pure integrator, type 0.
- $K_p = \lim_{s \to 0} G_e(s) = \dfrac{4K}{8-2K}$.
- $e_{ss} = \frac{1}{1+K_p} = \frac{8-2K}{8+2K}$.
- For $e_{ss} = 0 \rightarrow K = 4$.

Steady-State Error for Disturbances



T2050

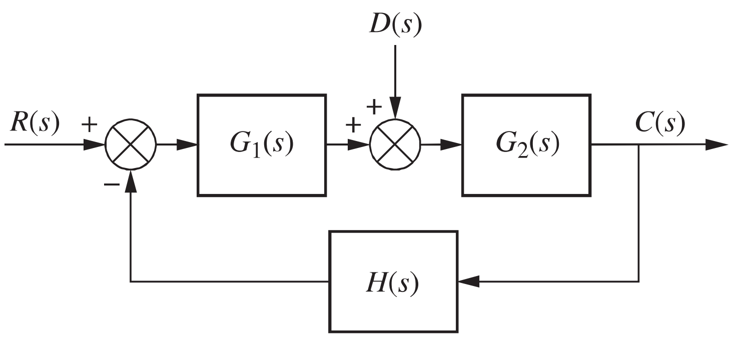
- $C(s) = E(s)G_1(s)G_2(s) + D(s)G_2(s)$: $\quad C(s) = R(s) - E(s)$

$\Rightarrow$ $E(s) = \frac{1}{1+G_1(s)G_2(s)}R(s) - \frac{G_2(s)}{1+G_1(s)G_2(s)}D(s)$

$$\boxed{e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{s}{1+G_1(s)G_2(s)}R(s) - \lim_{s\to 0} \frac{sG_2(s)}{1+G_1(s)G_2(s)}D(s)}$$

- $e_{ss,R} = \lim_{s\to 0} \frac{s}{1+G_1(s)G_2(s)}R(s)$, steady-state error due to $R(s)$,
- $e_{ss,D} = \lim_{s\to 0} \frac{sG_2(s)}{1+G_1(s)G_2(s)}D(s)$, steady state error due to disturbance.

Non-unity feedback control system with disturbance



T2045

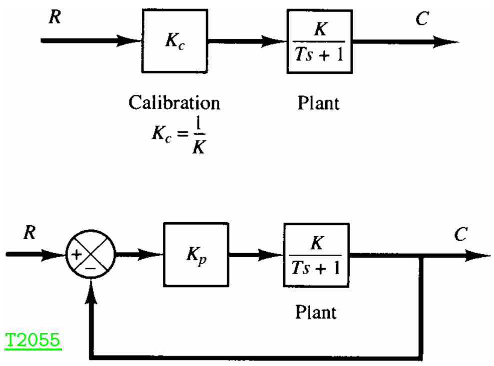
- $e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\left[\left\{1 - \frac{G_1(s)G_2(s)}{1+G_1(s)G_2(s)H(s)}\right\}R(s) - \left\{\frac{G_2(s)}{1+G_1(s)G_2(s)H(s)}\right\}D(s)\right]$
- For a step input and step disturbance:

$$e_{ss} = \left\{1 - \frac{\lim_{s\to 0} G_1(s)G_2(s)}{\lim_{s\to 0}[1+G_1(s)G_2(s)H(s)]}\right\} - \left\{\frac{\lim_{s\to 0} G_2(s)}{\lim_{s\to 0}[1+G_1(s)G_2(s)H(s)]}\right\}$$

- For zero error:
  - $\frac{\lim_{s\to 0} G_1(s)G_2(s)}{\lim_{s\to 0}[1+G_1(s)G_2(s)H(s)]} = 1$, and
  - $\frac{\lim_{s\to 0} G_2(s)}{\lim_{s\to 0}[1+G_1(s)G_2(s)H(s)]} = 0$.

Comparison: Open-loop & Feedback Controllers

$$e(t) = r(t) - c(t) \implies E(s) = R(s) - C(s)$$



Open-loop system, step input $(R(s) = 1/s)$:

- $E(s) = [1 - G_o(s)]R(s)$
- $e_{ss} = \lim_{s \to 0} sE(s) = \lim_{s \to 0} [1 - G_o(s)] \frac{1}{s} = 1 - G_o(0)$

Feedback system, step input $(R(s) = 1/s)$:

- $E(s) = \frac{1}{1+G(s)} R(s)$; $G(s) = \frac{K_p K}{Ts+1}$
- $e_{ss} = \lim_{s \to 0} sE(s) = \dfrac{1}{1 + K_p K}$
- If $K_p >> 1/K$, e_{ss} is small & fininte.

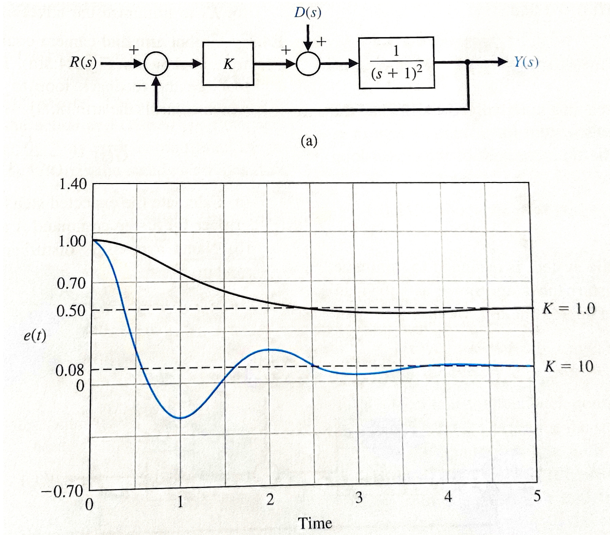
Open-loop system:

- By calibration, $K_c = 1/K \to KK_c = 1$: $\implies e_{ss} = 1 - G(0) = 0$
- Calibration may drift with time.
  - if $K \implies (K + \Delta K)$; & K =10, ΔK = 1:
  - $e_{ss} = 1 - K_c(K + \Delta K) = 1 - KK_c(1 + \frac{\delta K}{K}) = 1 - 1.1 = -0.1 = 10\%$
  - ⇒ Significant change.

Feedback system:

- Often $KK_p = 100 \to e_{ss} = \frac{1}{1+100} = 0.0099 = 9.9\%$.
  - if $K \implies (K + \Delta K)$; & K =10, ΔK = 1:
  - $e_{ss} = \frac{1}{1+K_p(K+\Delta K)} = \frac{1}{1+\frac{K_p}{K}(1+\frac{\Delta K}{K})} = \frac{1}{1+100(1.1)} = 9.0\%$
  - ⇒ Not significant change.

Example: ▷ Analyse Open-loop & Feedback Control Systems, $R(s) = 0$.

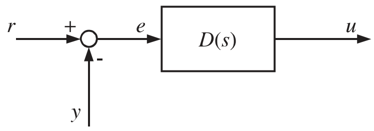


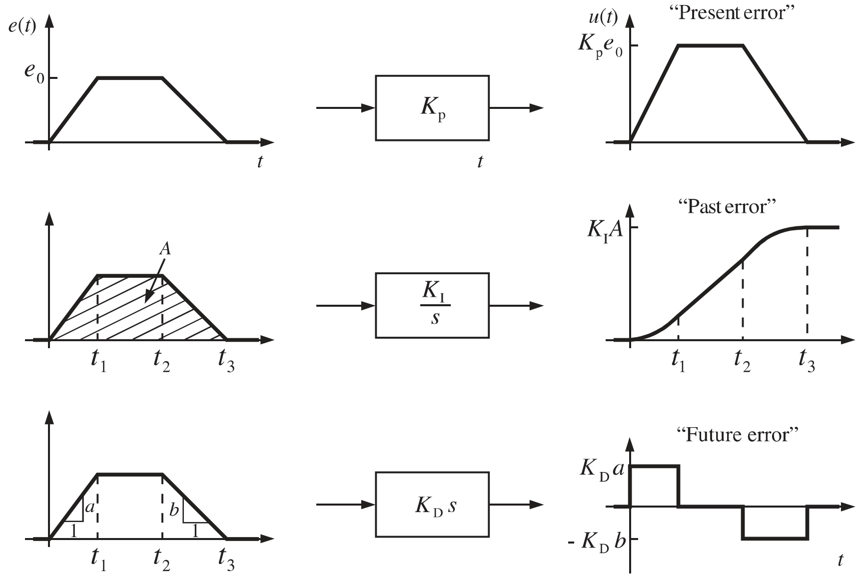
T2056

	Open Loop*	Closed Loop		
	$K = 1$	$K = 1$	$K = 8$	$K = 10$
Rise time (s) (10% to 90% of final value)	3.35	1.52	0.45	0.38
Percent overshoot (%)	0	4.31	33	40
Final value of $y(t)$ due to a disturbance, $T_d(s) = 1/s$	1.0	0.50	0.11	0.09
Percent steady-state error for unit step input	0	50%	11%	9%
Percent change in steady-state error due to 10% decrease in K	10%	5.3%	1.2%	0.9%

T2057

Input-Output Behaviour of Basic Control Actions

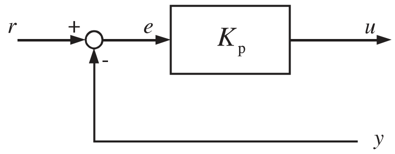




T2035

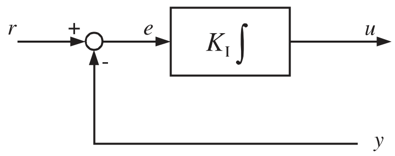
Basic Feed-back Control Actions

Proportional control (P)



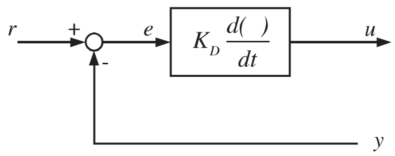
$$u(t) = K_p e(t)$$
$$D(s) = \frac{u(s)}{e(s)} = K_p$$

Integral control (I)



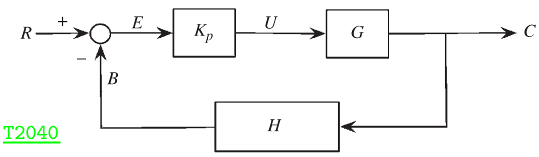
$$u(t) = K_I \int_0^t e(\tau) d\tau$$
$$D(s) = \frac{u(s)}{e(s)} = \frac{K_I}{s}$$

Derivative control (D)



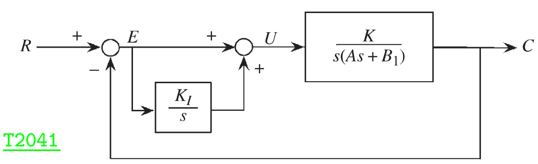
$$u(t) = K_D \frac{d}{dt}(e(t))$$
$$D(s) = \frac{u(s)}{e(s)} = K_D s$$

T2034



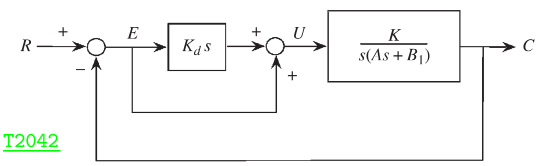
T2040

System including P control



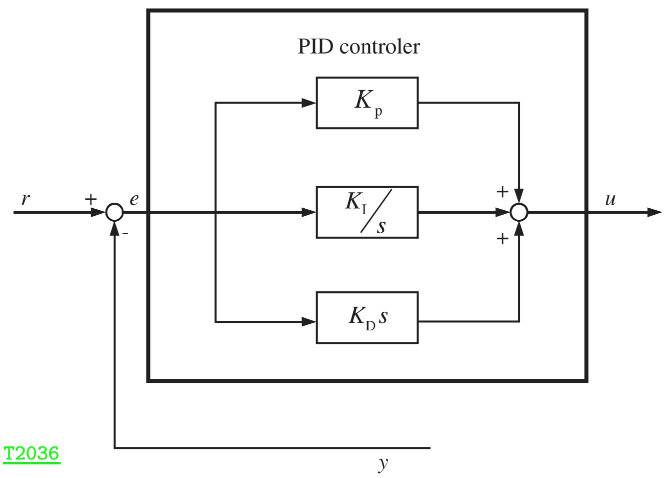
T2041

System including I control



T2042

System including D control



T2036

$$U(s) = (K_p + K_I \frac{1}{s} + K_D s)E(s)$$

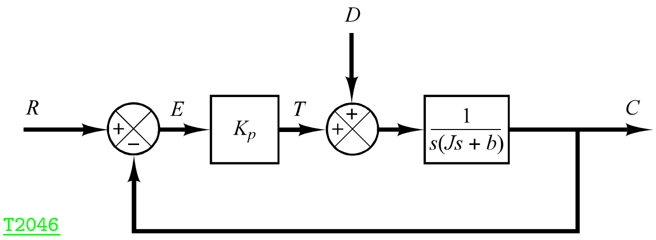
$$D(s) = K_p + K_I \frac{1}{s} + K_D s = K_p(1 + \frac{1}{T_i s} + T_D s)$$

$$K_i = \frac{K_P}{T_i}, \quad K_d = K_p T_D$$

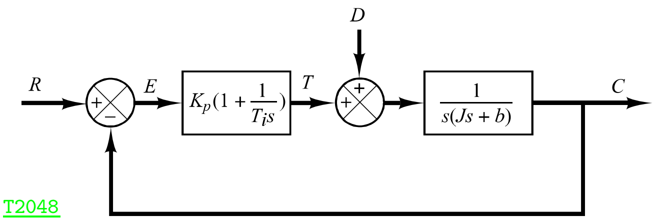
Response of P, PI Controllers

For a step disturbance, $D = 1/s$ & $H(s) = 1$:

$$e_{ss,D} = \frac{\lim_{s\to 0} G_2(s)}{\lim_{s\to 0}[1 + G_1(s)G_2(s)H(s)]} = -\frac{1}{\lim_{s\to 0} \frac{1}{G_2(s)} + \lim_{s\to 0} G_1(s)}$$



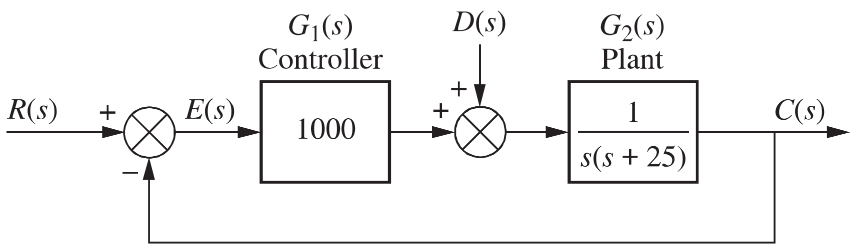
T2046



T2048

- For P: $e_{ss,D} = -1/K_p$
- For PI: $e_{ss,D} = 0$

Example: ▷ Steady-State Error Due to Step Disturbance



T2051

$[e_{ss,D} = 0.001]$