

DRAFT: State-Space Method

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1 Basics of State-space Method

- Formulation of State-space Equations
- Solution of State Equations

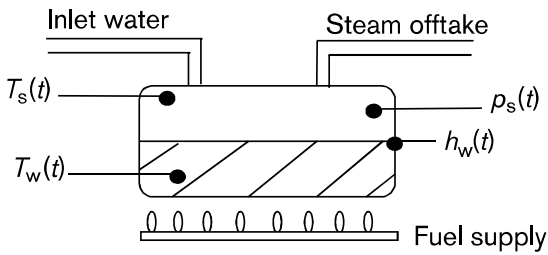
2 Design by State-space

- Controllability
- Observability



What is a state variable?

Steam Boiler



$$\bullet \mathbf{x}_{sb}(t) = [T_s(t), p_s(t), T_w(t), h_w(t)]^T$$

State variables

$T_s(t)$	Temperature of steam
$p_s(t)$	Pressure of steam
$T_w(t)$	Temperature of water
$h_w(t)$	Level of water

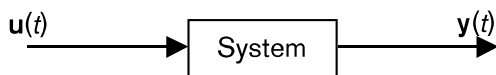
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$$\mathbf{x}_{sb}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} T_s(t) \\ p_s(t) \\ T_w(t) \\ h_w(t) \end{bmatrix}$$

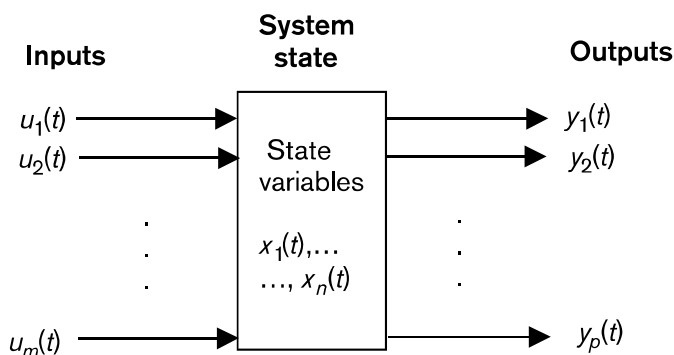
The state or set of system variables provides us with the status of a particular system at any instant in time.



State variable representation



General system block diagram.



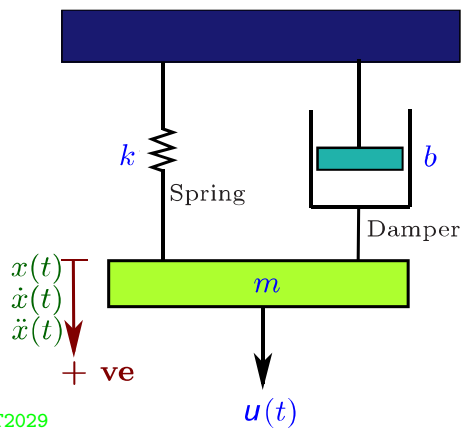
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Inputs, outputs and state variables.

- 1 Define system equations,
- 2 Identify system inputs, outputs and states,
- 3 Rewrite new system equations in standard state variable vector-matrix notation.



State-space Method: Mechanical System



$$\bullet \quad m \frac{d^2 y}{dt^2} + b \frac{dy}{dt} + ky = u$$

$$\Rightarrow \frac{d^2 y}{dt^2} + \frac{b}{m} \frac{dy}{dt} + \frac{k}{m} y = \frac{u}{m}$$

- State variables, $x_1(t)$ and $x_2(t)$:

$$x_1(t) = y(t)$$

$$x_2(t) = \dot{y}(t)$$

T2029

- State Equation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u \Rightarrow \dot{x} = Ax(t) + Bu(t)$$

- Output Equation: $y = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow y = Cx(t) + Du(t)$

$$\text{where, } C = [1 \quad 0], D = 0$$

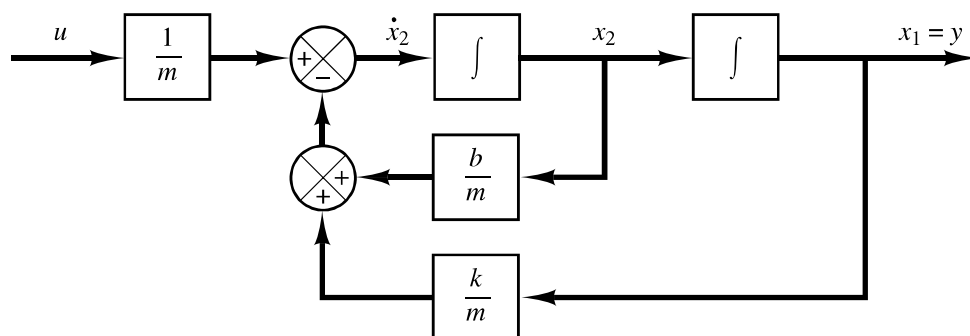


$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$y = \mathbf{C}\mathbf{x} + Du$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}, \quad \mathbf{C} = [1 \quad 0], \quad D = 0$$

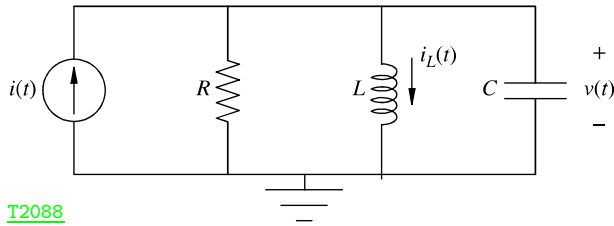
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T2086



Example: Parallel RLC Circuit



T2088

State Differential Equation

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & -\frac{1}{RC} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{C} \end{bmatrix} u(t)$$

Algebraic Output Equation

$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} u(t)$$

T2089

• State variables:

- ① $x_1(t) = i_L(t)$
- ② $x_2(t) = v(t)$

• Governing equations:

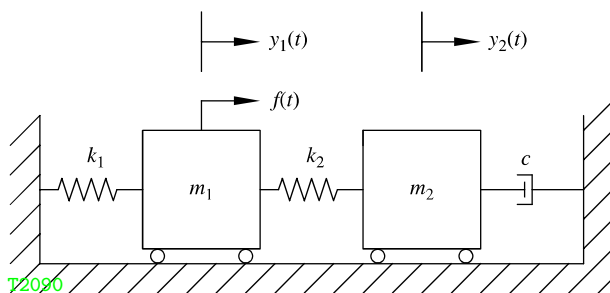
- ① $x_2(t) = L\dot{x}_1(t)$
- ② $\frac{x_2(t)}{R} + x_1(t) + C\dot{x}_2(t) = u(t)$

• Rearranged equations:

- ① $\dot{x}_1(t) = \frac{x_2(t)}{L}$
- ② $\dot{x}_2(t) = -\frac{x_1(t)}{C} - \frac{x_2(t)}{RC} + \frac{u(t)}{C}$



Example: Translational Mechanical System



T2090

State Differential Equation

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ -\frac{k_1}{m_1} & \frac{k_2}{m_1} & 0 & 0 \\ 0 & -\frac{k_2}{m_2} & 0 & -\frac{c}{m_2} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{m_1} \\ 0 \end{bmatrix} u(t)$$

Algebraic Output Equation

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u(t)$$

T2091

- ① $x_1(t) = y_1(t)$
- ② $x_2(t) = y_2(t) - y_1(t)$
- ③ $x_3(t) = \dot{y}_1(t)$
- ④ $x_4(t) = \dot{y}_2(t)$

$$m_1 \ddot{y}_1(t) + k_1 y_1(t) - k_2 [y_2(t) - y_1(t)] = f(t)$$

$$m_2 \ddot{y}_2(t) + c y_2(t) + k_2 [y_2(t) - y_1(t)] = 0$$



State variable notation summary

Given a system of

m inputs

n states

r outputs

the full state space system is given by

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

where $\mathbf{A}$ (size $n \times n$) is the system matrix

$\mathbf{B}$ (size $n \times m$) is the input matrix

$\mathbf{C}$ (size $r \times n$) is the output matrix

$\mathbf{D}$ (size $r \times m$) is the direct feedthrough matrix

The matrix $\mathbf{D}$ represents any direct connections between the input and the output. However, in many simple cases, such as the trailer suspension example, the $\mathbf{D}$ matrix is zero.

T2084

The step of identifying the number of states (n), inputs (m) and outputs (r) automatically sets up the size of the ABCD matrices to be filled:

$$\dot{\mathbf{x}}(t) = \mathbf{A}_{n \times n} \mathbf{x}(t) + \mathbf{B}_{n \times m} \mathbf{u}(t)$$

$$\mathbf{y}(t) = \mathbf{C}_{r \times n} \mathbf{x}(t) + \mathbf{D}_{r \times m} \mathbf{u}(t)$$



T2085

- State equation and output equation:

- 1 $\dot{x} = f(x, u, t)$

- 2 $y = g(x, u, t)$

- Linearised state equation and output equation:

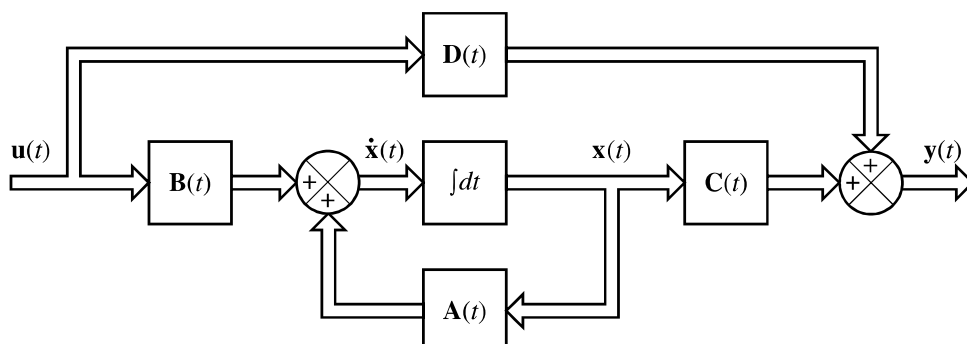
- 1 $\dot{x} = A(t)x(t) + B(t)u(t)$

- 2 $y = C(t)x(t) + D(t)u(t)$

- Time-invariant linearised state equation and output equation:

- 1 $\dot{x} = Ax(t) + Bu(t)$

- 2 $y = Cx(t) + Du(t)$



T2033



Example: State-space representation of a DE

$$\ddot{y}(t) + a_2 \dot{y}(t) + a_1 y(t) + a_0 y(t) = b_0 u(t): H(s) = \frac{b_0}{s^3 + a_2 s^2 + a_1 s + a_0}$$

- State-variables, $\mathbf{x}(t) = [x_1(t), x_2(t), x_3(t)]^T$

- 1 $x_1(t) = y(t)$

- 2 $x_2(t) = \dot{y}(t) = \dot{x}_1(t)$

- 3 $x_3(t) = \ddot{y}(t) = \ddot{x}_1(t) = \dot{x}_2(t)$

- $x_3(t) = -a_0 x_1(t) - a_2 x_2(t) - a_3 x_3(t) + b_0 u(t)$

State Differential Equation

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ b_0 \end{bmatrix} u(t)$$

Algebraic Output Equation

$$y(t) = [1 \quad 0 \quad 0] \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + [0]u(t)$$

T2092



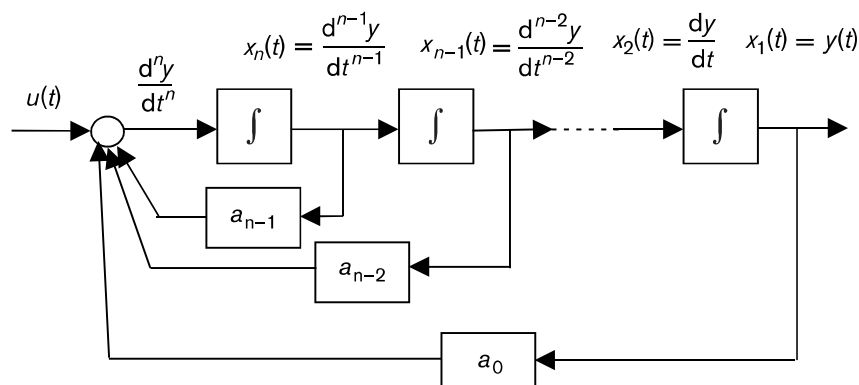
n-th order Differential Equation (DE)

$$\frac{d^n y(t)}{dt^n} + a_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}} + \dots + a_2 \frac{d^2 y(t)}{dt^2} + a_1 \frac{dy(t)}{dt} + a_0 y(t) = b_0 u(t)$$

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b_0 \end{bmatrix}$$

$$C = [1 \quad 0 \quad 0 \quad \dots \quad 0] \quad D = [0]$$

T2093



T2095

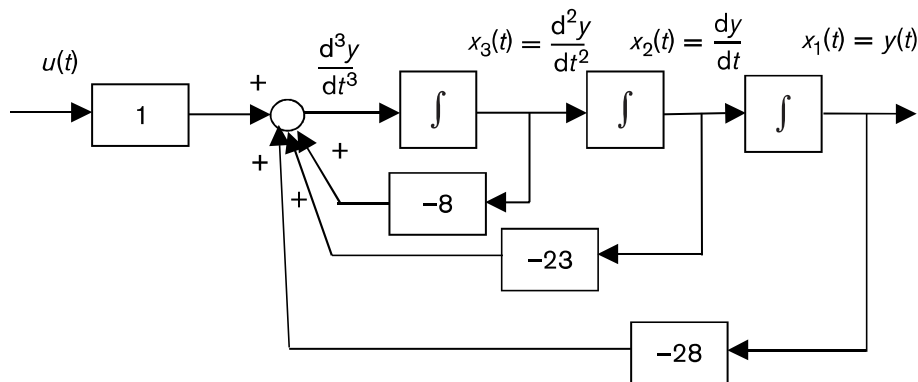


Example: $\triangleright Y(s) = \frac{1}{s^3 + 8s^2 + 23s + 28} U(s) \rightarrow$
 $\frac{d^3 y(t)}{dt^3} + 8 \frac{d^2 y(t)}{dt^2} + 23 \frac{dy(t)}{dt} + 28y(t) = u(t)$

T2097

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -28 & -23 & -8 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \mathbf{x}$$

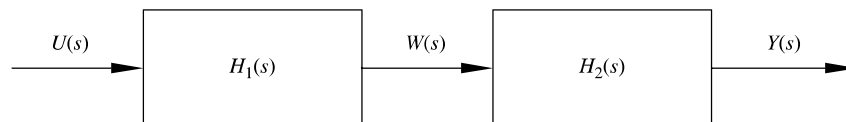


T2096



Example: $\triangleright H(s) = \frac{b_2 s^2 + b_1 s + b_o}{s^3 + a_2 s^2 + a_1 s + a_o} = H_1(s) H_2(s)$

T2094



- $W(s) = H_1(s) U(s) = \frac{1}{s^3 + a_2 s^2 + a_1 s + a_o} U(s)$
- $Y(s) = H_2(s) W(s) = (b_2 s^2 + b_1 s + b_o) W(s)$
- $\ddot{w}(t) + a_2 \dot{w}(t) + a_1 w(t) + a_o w(t) = u(t)$
- $y(t) = b_2 \ddot{w}(t) + b_1 \dot{w}(t) + b_o w(t)$

T2103

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$w(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + [0] u(t)$$



$$\begin{aligned} y(t) &= b_o w(t) + b_1 \dot{w}(t) + b_2 \ddot{w}(t) \\ &= b_o x_1(t) + b_1 x_2(t) + b_2 x_3(t) \end{aligned}$$

T2103

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$w(t) = \begin{bmatrix} b_o & b_1 & b_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + [0] u(t)$$



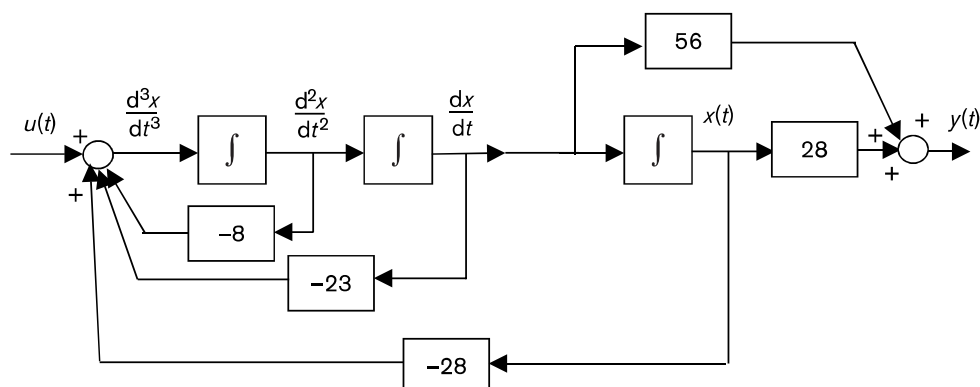
Example: $\triangleright Y(s) = \frac{56s+28}{s^3+8s^2+23s+28} U(s)$

T2097

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -28 & -23 & -8 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

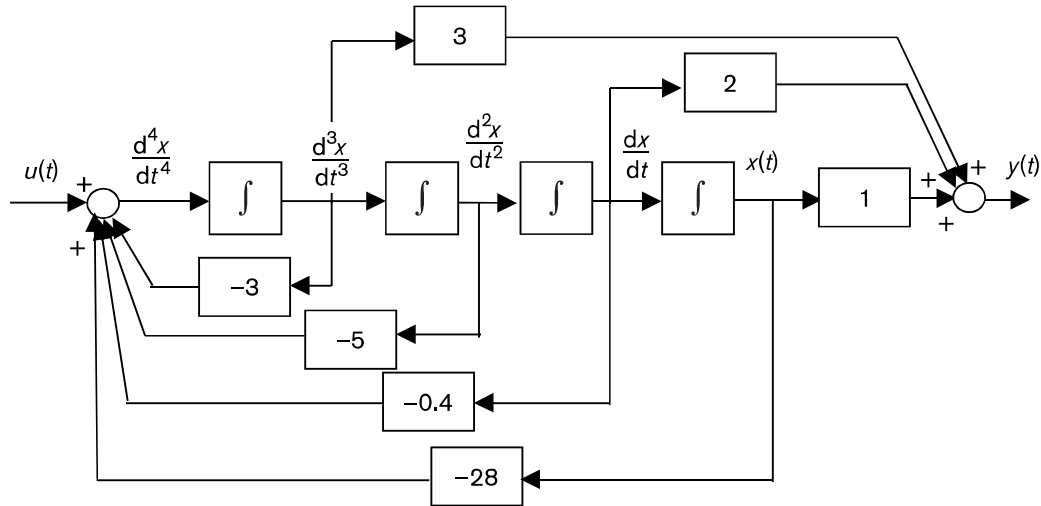
$$y(t) = \begin{bmatrix} 28 & 56 & 0 \end{bmatrix} \mathbf{x}$$

T2098



Example: $\triangleright Y(s) = \frac{3s^3 + 2s + 1}{s^4 + 3s^3 + 5s^2 + 0.4s + 28} U(s)$

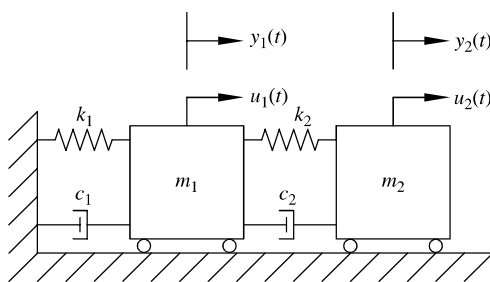
$$y(t) = \begin{bmatrix} 1 & 2 & 0 & 3 \end{bmatrix} \mathbf{x}$$



T2099



Example: Multiple-input, multiple-output system

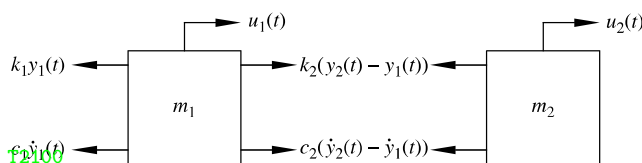


$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-(k_1 + k_2)}{m_1} & \frac{-(c_1 + c_2)}{m_1} & \frac{k_2}{m_1} & \frac{c_2}{m_1} \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & \frac{c_2}{m_2} & \frac{-k_2}{m_2} & \frac{-c_2}{m_2} \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ \frac{1}{m_1} & 0 \\ 0 & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix}$$

T2101

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

T2102



T2103

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$m_1 \ddot{y}_1(t) + (c_1 + c_2) \dot{y}_1(t) + (k_1 + k_2) y_1(t) - c_2 \dot{y}_2(t) - k_2 y_2(t) = u_1(t)$$

$$m_2 \ddot{y}_2(t) + c_2 \dot{y}_2(t) + k_2 y_2(t) - c_2 \dot{y}_1(t) - k_2 y_1(t) = u_2(t)$$



State-space Method to TF

- $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$
- $y = \mathbf{C}\mathbf{x} + \mathbf{D}u$
- $sX(s) - x(0) = AX(s) + BU(s)$: taking Laplace Transform
- $Y(s) = CX(s) + DU(s)$: taking Laplace Transform
- ⇒ $sX(s) = AX(s) + BU(s)$: $x(0) = 0$
- $(sI - A)X(s) = BU(s)$
- ⇒ $X(s) = (sI - A)^{-1}BU(s)$: multiplying $(sI - A)^{-1}$ to both sides
- $Y(s) = [C(sI - A)^{-1}B + D] U(s)$

$$G(s) = C(sI - A)^{-1}B + D$$



Example: Mechanical System

- State Equation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u \Rightarrow \dot{\mathbf{x}} = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$

- Output Equation: $y = [1 \ 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow y = \mathbf{C}\mathbf{x}(t) + \mathbf{D}u(t)$

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D$$

$$\begin{aligned} &= [1 \ 0] \left\{ \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \right\}^{-1} \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} + 0 \quad G(s) = [1 \ 0] \frac{1}{s^2 + \frac{b}{m}s + \frac{k}{m}} \begin{bmatrix} s + \frac{b}{m} & 1 \\ -\frac{k}{m} & s \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} \\ &\stackrel{\text{T2104}}{=} [1 \ 0] \begin{bmatrix} s & -1 \\ \frac{k}{m} & s + \frac{b}{m} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} \quad \stackrel{\text{T2105}}{=} \frac{1}{ms^2 + bs + k} \end{aligned}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$



Example: ▷ Obtain TF from State-space equations:

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{C} = [1 \ 0 \ 0], \quad D = 0$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \quad G(s) = [1 \ 0 \ 0] \begin{bmatrix} s+1 & -1 & 0 \\ 0 & s+1 & -1 \\ 0 & 0 & s+2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$y = [1 \ 0 \ 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$G(s) = C(sI - A)^{-1}B + D$$

$$= [1 \ 0 \ 0] \begin{bmatrix} \frac{1}{s+1} & \frac{1}{(s+1)^2} & \frac{1}{(s+1)^2(s+2)} \\ 0 & \frac{1}{s+1} & \frac{1}{(s+1)(s+2)} \\ 0 & 0 & \frac{1}{s+2} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{(s+1)^2(s+2)} = \frac{1}{s^3 + 4s^2 + 5s + 2}$$



Example: ▷ Obtain TF from State-space equations:

$$\mathbf{G}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 6.5 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$G(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s+1 & 1 \\ -6.5 & s \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \frac{1}{s^2 + s + 6.5} \begin{bmatrix} s & -1 \\ 6.5 & s+1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \frac{1}{s^2 + s + 6.5} \begin{bmatrix} s-1 & s \\ s+7.5 & 6.5 \end{bmatrix}$$

$$Y(s) =$$

$$[C(sI - A)^{-1}B + D] U(s)$$

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} \frac{s-1}{s^2 + s + 6.5} & \frac{s}{s^2 + s + 6.5} \\ \frac{s+7.5}{s^2 + s + 6.5} & \frac{6.5}{s^2 + s + 6.5} \end{bmatrix} \begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix}$$



Solution of Homogeneous State Equations

- $\dot{x}(t) = Ax(t)$
- $sX(s) - x(0) = AX(s)$
- $(sI - A)X(s) = x(0)$
- ⇒ $X(s) = (sI - A)^{-1}x(0)$
- ⇒ $x(t) = L^{-1}[(sI - A)^{-1}]x(0) = \phi(t)x(0)$
- State-transition matrix, $\phi(t) = L^{-1}[(sI - A)^{-1}]$
- $(sI - A) = \frac{I}{s} + \frac{A}{s^2} + \frac{A^2}{s^3} + \dots$
- $\phi(t) = L^{-1}[(sI - A)^{-1}] = I + At + \frac{A^2 t^2}{2!} + \frac{A^3 t^3}{3!} + \dots = e^{At}$

$$x(t) = \phi(t)x(0) = e^{At}x(0)$$



Example: ▷ Obtain the state-transition matrix of the following system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- $\phi(t) = e^{At} = L^{-1}[(sI - A)^{-1}]$
- $(sI - A) = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$

$$\begin{aligned} (s\mathbf{I} - \mathbf{A})^{-1} &= \frac{1}{(s+1)(s+2)} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} \\ &= \begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix} \end{aligned}$$

T2111

$$\begin{aligned} \Phi(t) &= e^{At} = \mathcal{L}^{-1}[(s\mathbf{I} - \mathbf{A})^{-1}] \\ &= \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} \end{aligned}$$

T2112



Solution of Non-Homogeneous State Equations

- $\dot{x}(t) = Ax(t) + Bu(t)$
- $sX(s) - x(0) = AX(s) + BU(s)$
- $(sI - A)X(s) = x(0) + BU(s)$
- ⇒ $X(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}BU(s) =$
 $L[e^{At}]x(0) + L[e^{At}]BU(s)$
- ⇒ $x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$
- If initial time is given by t_0 instead of 0:

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^t e^{A(t-\tau)}Bu(\tau)d\tau$$



Example: ▷ Obtain the state-transition matrix of the following system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$\begin{aligned} \Phi(t) &= e^{At} = \mathcal{L}^{-1}[(sI - A)^{-1}] \\ &= \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} \end{aligned}$$

T2112

The response to the unit-step input is then obtained as:

$$\mathbf{x}(t) = e^{At}\mathbf{x}(0) + \int_0^t \begin{bmatrix} 2e^{-(t-\tau)} - e^{-2(t-\tau)} & e^{-(t-\tau)} - e^{-2(t-\tau)} \\ -2e^{-(t-\tau)} + 2e^{-2(t-\tau)} & -e^{-(t-\tau)} + 2e^{-2(t-\tau)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} d\tau$$

T2113

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

If the initial state is zero, or $x(0)=0$, then $x(t)$ can be simplified to:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

T2114



Controllability

If an input to a system can be found that takes every state variable from a desired initial state to a desired final state, the system is said to be **controllable**; otherwise, the system is **uncontrollable**.

- An n^{th} -order plant whose state equation:

$$\dot{x} = Ax + Bu$$

- Completely controllable if the matrix:

$$P = [B \quad AB \quad A^2B \quad \dots \quad A^{n-1}B]$$

is of rank n , where P is called the **controllability matrix**.



Example: ▷ Investigate controllability.

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} u(t)$$

$$B = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 7 \end{bmatrix}$$

$$A^2B = A(AB) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ 7 \end{bmatrix} = \begin{bmatrix} -3 \\ 7 \\ -15 \end{bmatrix}$$

- $|P| = 0$, thus rank of P is less than 3. So, not-controllable.

$$\begin{aligned} P &= [B \quad AB \quad A^2B] \\ &= \begin{bmatrix} 0 & 1 & -3 \\ 1 & -3 & 7 \\ -3 & 7 & -15 \end{bmatrix} \end{aligned}$$

T2116



Example: ▷ Investigate controllability.

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} b_0 & b_1 & b_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

T2117

$$B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad AB = \begin{bmatrix} 0 \\ 1 \\ -a_2 \end{bmatrix}$$

- $|P| = -1 \neq 0$. So, controllable.

$$A^2B = \begin{bmatrix} 1 \\ -a_2 \\ a_2^2 - a_1 \end{bmatrix}$$

$$P = \begin{bmatrix} B & AB & A^2B \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -a_2 \\ 1 & -a_2 & a_2^2 - a_1 \end{bmatrix}$$

T2118



Observability

If the initial-state vector, $x(t_0)$, can be found from $u(t)$ and $y(t)$ measured over a finite interval of time from t_0 , the system is said to be observable; otherwise the system is said to be unobservable.

- An n^{th} -order plant whose state equation & output equation:

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

- Completely observable if the matrix:

$$Q = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

is of rank n , where Q is called the **observability matrix**.



Example: ▷ Investigate observability.

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

$$y(t) = \begin{bmatrix} 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

T2119

$$C = \begin{bmatrix} 0 & 1 & -3 \end{bmatrix}$$

$$CA = \begin{bmatrix} 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 7 \end{bmatrix}$$

$$CA^2 = (CA)A = \begin{bmatrix} 1 & -3 & 7 \end{bmatrix} \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix} = \begin{bmatrix} -3 & 7 & -15 \end{bmatrix}$$

$$Q = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -3 \\ 1 & -3 & 7 \\ -3 & 7 & -15 \end{bmatrix}$$

T2120

- $|Q| = 0$, thus rank of Q is less than 3. So, non-observable.

