State-Space Analysis

Dr. Md. Zahurul Haq, *Ph.D., CEA, FBSME, FIEB*

Professor
Department of Mechanical Engineering
Bangladesh University of Engineering & Technology (BUET)
Dhaka-1000, Bangladesh

http://zahurul.buet.ac.bd/

RME 3204: Control Systems Design
Department of Robotics and Mechatronics Engineering,
University of Dhaka

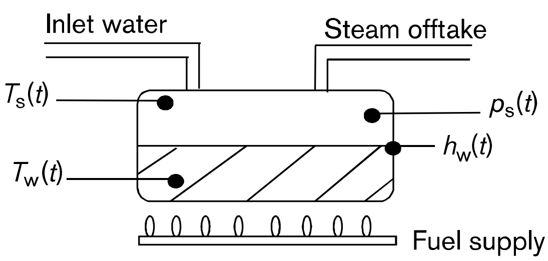
http://zahurul.buet.ac.bd/RME3204/

Motivation for State-space Method

- Classical, or frequency-domain approach have the primary disadvantage of its limited applicability: It can be applied only to linear, time-invariant systems or system that can be approximated as such.
- A major advantage of frequency-domain techniques is that they rapidly provide stability and transient response information.
- State-space, modern, or time-domain approach is a unified method for analysing and designing a wide range of systems:
  - Non-linear systems that have backlash, saturation, and dead zone.
  - Time-varying systems e.g. missiles with varying fuel levels.
  - Multiple-input, multiple-output (MIMO) systems (such as a vehicle with input direction and input velocity yielding an output direction and an output velocity) can be compactly represented in state space with a model similar in form and complexity to that used for single-input, single-output (SISO) systems.

What is a state variable?

Steam Boiler



State variables	
$T_s(t)$	Temperature of steam
$p_s(t)$	Pressure of steam
$T_w(t)$	Temperature of water
$h_w(t)$	Level of water

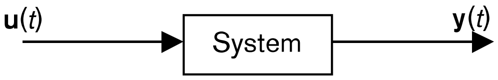
T2082

The state or set of system variables provides us with the status of a particular system at any instant in time.

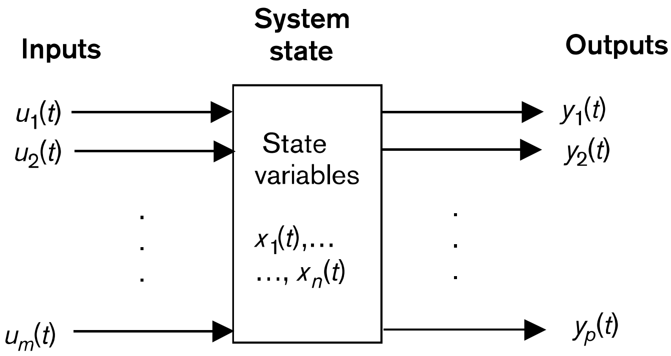
- $\mathrm{x}_{sb}(t) = [T_s(t), p_s(t), T_w(t), h_w(t)]^T$

$$\mathrm{x}_{sb}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} T_s(t) \\ p_s(t) \\ T_w(t) \\ h_w(t) \end{bmatrix}$$

State variable representation



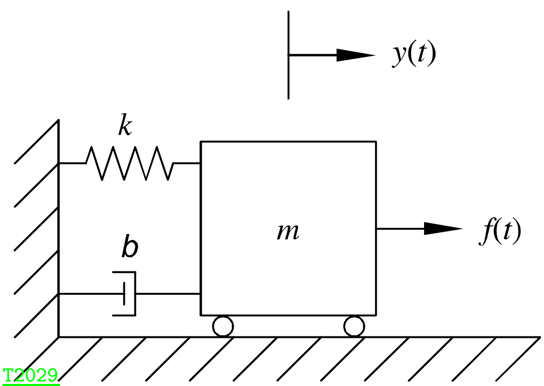
General system block diagram.



T2083 Inputs, outputs and state variables.

1. Define system equations,
2. Identify system inputs, outputs and states,
3. Rewrite new system equations in standard state variable vector-matrix notation.

State-space Method: Mechanical System



- $m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = u$

$\Rightarrow \frac{d^2y}{dt^2} + \frac{b}{m}\frac{dy}{dt} + \frac{k}{m}y = \frac{u}{m}$

- State variables, $x_1(t)$ and $x_2(t)$:

$$x_1(t) = y(t)$$

$$x_2(t) = \dot{y}(t)$$

- State Equation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u \Rightarrow \dot{x} = A\mathrm{x(t)} + B\mathrm{u(t)}$$

- Output Equation: $y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow y = C\mathrm{x(t)} + D\mathrm{u(t)}$

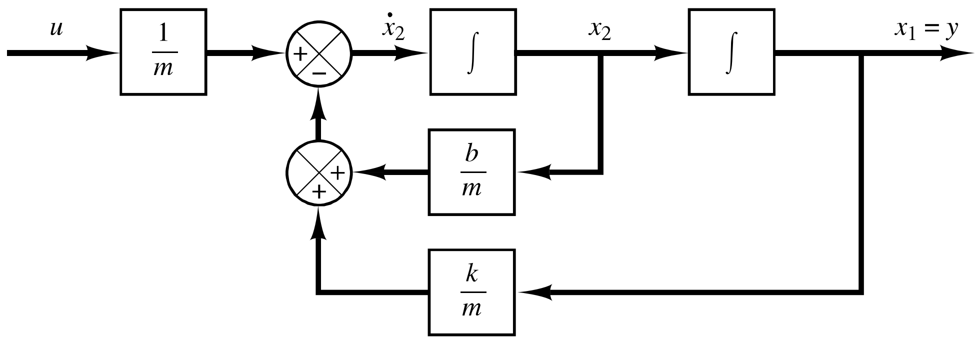
where, $C = \begin{bmatrix} 1 & 0 \end{bmatrix}, D = 0$

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$y = \mathbf{C}\mathbf{x} + Du$$

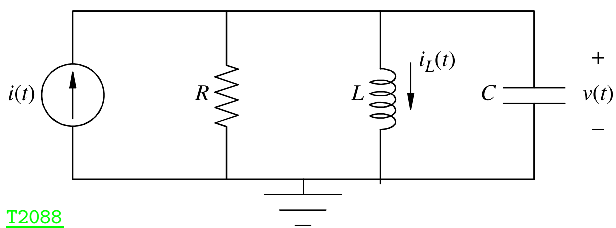
$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{b}{m} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ \dfrac{1}{m} \end{bmatrix}, \quad \mathbf{C} = [1 \quad 0], \quad D = 0$$

T2087



T2086

Example: Parallel RLC Circuit



T2088

State Differential Equation

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & \dfrac{1}{L} \\ -\dfrac{1}{C} & -\dfrac{1}{RC} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{C} \end{bmatrix} u(t)$$

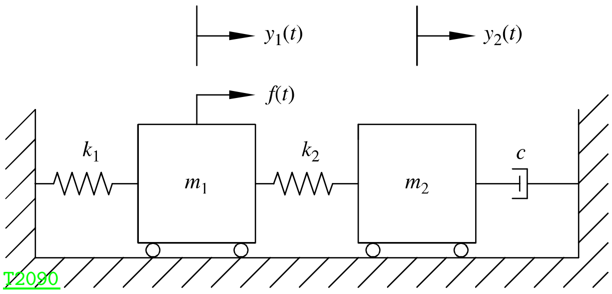
Algebraic Output Equation

$$y(t) = [0 \quad 1] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + [0]u(t)$$

T2089

- State variables:
  1. $x_1(t) = i_L(t)$
  2. $x_2(t) = v(t)$
- Governing equations:
  1. $x_2(t) = L\dot{x}_1(t)$
  2. $\frac{x_2(t)}{R} + x_1(t) + C\dot{x}_2(t) = u(t)$
- Rearranged equations:
  1. $\dot{x}_1(t) = \frac{x_2(t)}{L}$
  2. $\dot{x}_2(t) = -\frac{x_1(t)}{C} - \frac{x_2(t)}{RC} + \frac{u(t)}{C}$

Example: Translational Mechanical System



T2090

State Differential Equation

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{x}_3(t)\\ \dot{x}_4(t)\end{bmatrix} = \begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & -1 & 1\\ -\dfrac{k_1}{m_1} & \dfrac{k_2}{m_1} & 0 & 0\\ 0 & \dfrac{-k_2}{m_2} & 0 & -\dfrac{c}{m_2}\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\\ x_4(t)\end{bmatrix} + \begin{bmatrix}0\\ 0\\ \dfrac{1}{m_1}\\ 0\end{bmatrix}u(t)$$

Algebraic Output Equation

$$\begin{bmatrix}y_1(t)\\ y_2(t)\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 0\\ 1 & 1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\\ x_4(t)\end{bmatrix} + \begin{bmatrix}0\\ 0\end{bmatrix}u(t)$$

T2091

1. $x_1(t) = y_1(t)$
2. $x_2(t) = y_2(t) - y_1(t)$
3. $x_3(t) = \dot{y}_1(t)$
4. $x_4(t) = \dot{y}_2(t)$

$$m_1\ddot{y}_1(t) + k_1 y_1(t) - k_2[y_2(t) - y_1(t)] = f(t)$$

$$m_2\ddot{y}_2(t) + c y_2(t) + k_2[y_2(t) - y_1(t)] = 0$$

State variable notation summary

Given a system of

m inputs
n states
r outputs

the full state space system is given by

$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$

$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$

where **A** (size $n \times n$) is the system matrix
B (size $n \times m$) is the input matrix
C (size $r \times n$) is the output matrix
D (size $r \times m$) is the direct feedthrough matrix

The matrix **D** represents any direct connections between the input and the output. However, in many simple cases, such as the trailer suspension example, the **D** matrix is zero.

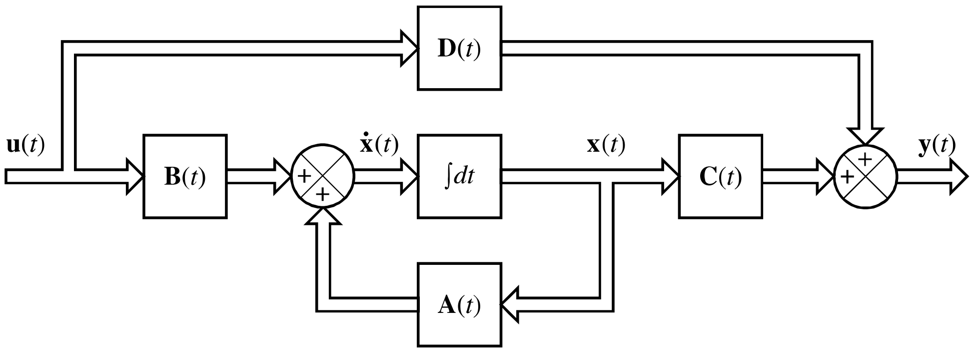
T2084

The step of identifying the number of states (n), inputs (m) and outputs (r) automatically sets up the size of the ABCD matrices to be filled:

$$\dot{\mathbf{x}}(t) = \mathbf{A}_{n\times n}\mathbf{x}(t) + \mathbf{B}_{n\times m}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}_{r\times n}\mathbf{x}(t) + \mathbf{D}_{r\times m}\mathbf{u}(t)$$

T2085

- State equation and output equation:
  1. $\dot{x} = f(x, u, t)$
  2. $y = g(x, u, t)$
- Linearised state equation and output equation:
  1. $\dot{x} = A(t)x(t) + B(t)u(t)$
  2. $y = C(t)x(t) + D(t)u(t)$
- Time-invariant linearised state equation and output equation:
  1. $\dot{x} = Ax(t) + Bu(t)$
  2. $y = Cx(t) + Du(t)$



T2033

Example: State-space representation of a DE

$$\dddot{y}(t) + a_2\ddot{y}(t) + a_1\dot{y}(t) + a_o y(t) = b_o u(t): \; H(s) = \frac{b_o}{s^3 + a_2 s^2 + a_1 s + a_o}$$

- State-variables, $x(t) = [x_1(t), \; x_2(t), \; x_3(t)]^T$
  1. $x_1(t) = y(t)$
  2. $x_2(t) = \dot{y}(t) = \dot{x}_1(t)$
  3. $x_3(t) = \ddot{y}(t) = \ddot{x}_1(t) = \dot{x}_2(t)$
- $x_3(t) = -a_o x_1(t) - a_2 x_2(t) - a_3 x_3(t) + b_o u(t)$

State Differential Equation

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ b_0 \end{bmatrix} u(t)$$

Algebraic Output Equation

$$y(t) = [1 \quad 0 \quad 0] \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + [0]u(t)$$

T2092

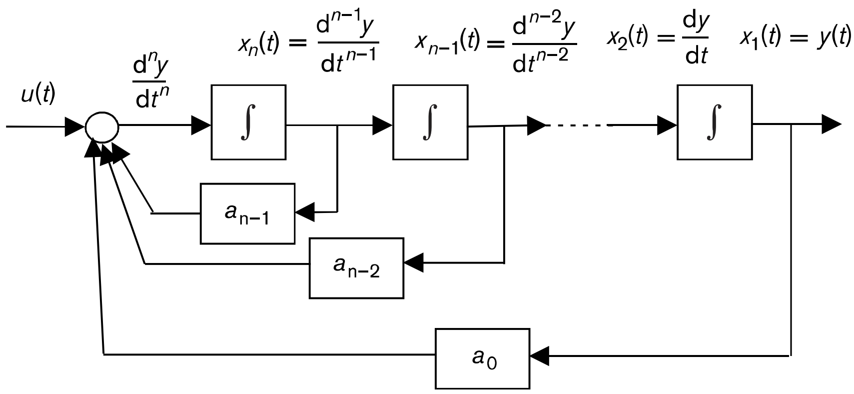
n-th order Differential Equation (DE)

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \cdots + a_2\frac{d^2y(t)}{dt^2} + a_1\frac{dy(t)}{dt} + a_0 y(t) = b_o u(t)$$

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b_0 \end{bmatrix}$$

T2093

$$C = [1 \quad 0 \quad 0 \quad \cdots \quad 0] \qquad D = [0]$$



T2095

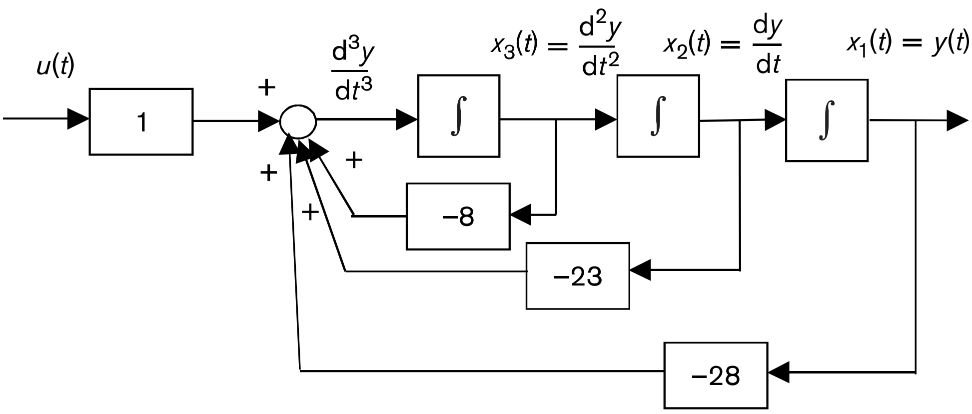
Example: ▷ $Y(s) = \frac{1}{s^3+8s^2+23s+28}U(s)$: $\longrightarrow$

$$\frac{d^3y(t)}{dt^3} + 8\frac{d^2y(t)}{dt^2} + 23\frac{d^y(t)}{dt} + 28y(t) = u(t)$$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -28 & -23 & -8 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

T2097

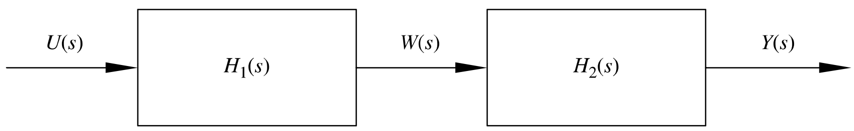
$$y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \mathbf{x}$$



T2096

Example: $\triangleright\ H(s) = \frac{b_2 s^2 + b_1 s + b_o}{s^3 + a_2 s^2 + a_1 s + a_o} = H_1(s) H_2(s)$

T2094



- $W(s) = H_1(s) U(s) = \frac{1}{s^3 + a_2 s^2 + a_1 s + a_o} U(s)$
- $Y(s) = H_2(s) W(s) = (b_2 s^2 + b_1 s + b_o) W(s)$
- $\dddot{w}(t) + a_2 \ddot{w}(t) + a_1 \dot{w}(t) + a_o w(t) = u(t)$
- $y(t) = b_2 \ddot{w}(t) + b_1 \dot{w}(t) + b_o w(t)$

T2103

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$w(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + [0] u(t)$$

$$\begin{aligned} y(t) &= b_o w(t) + b_1 \dot{w}(t) + b_2 \ddot{w}(t) \\ &= b_o x_1(t) + b_1 x_2(t) + b_2 x_3(t) \end{aligned}$$

T2103

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

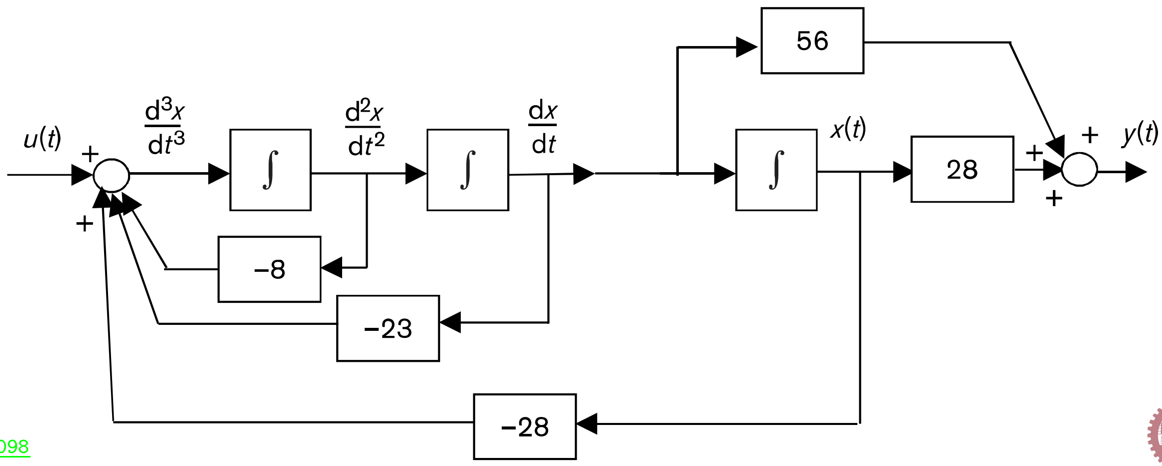
$$w(t) = \begin{bmatrix} b_0 & b_1 & b_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + [0] u(t)$$

Example: ▷ $Y(s) = \frac{56s+28}{s^3+8s^2+23s+28}U(s)$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -28 & -23 & -8 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

T2097

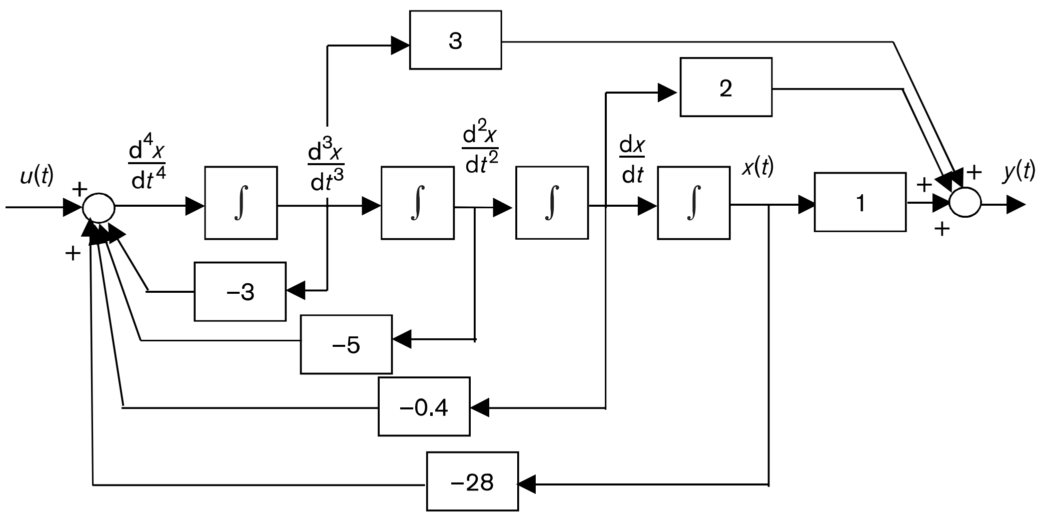
$$y(t) = \begin{bmatrix} 28 & 56 & 0 \end{bmatrix} \mathrm{x}$$



T2098

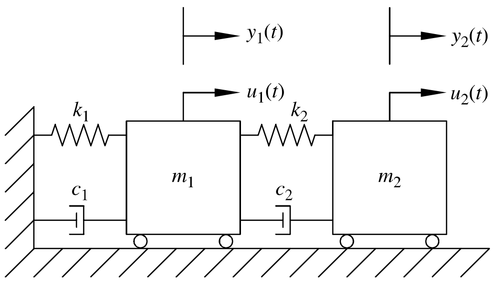
Example: ▷ $Y(s) = \frac{3s^3+2s+1}{s^4+3s^3+5s^2+0.4s+28}U(s)$

$$y(t) = \begin{bmatrix} 1 & 2 & 0 & 3 \end{bmatrix} \mathrm{x}$$



T2099

Example: Multiple-input, multiple-output system

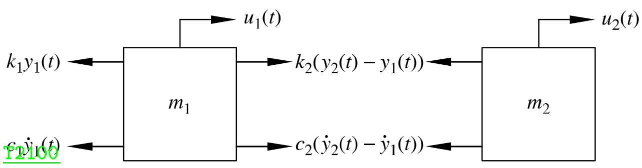


$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-(k_1+k_2)}{m_1} & \frac{-(c_1+c_2)}{m_1} & \frac{k_2}{m_1} & \frac{c_2}{m_1} \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & \frac{c_2}{m_2} & \frac{-k_2}{m_2} & \frac{-c_2}{m_2} \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ \frac{1}{m_1} & 0 \\ 0 & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix}$$

T2101

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

T2102



$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$m_1\ddot{y}_1(t) + (c_1 + c_2)\dot{y}_1(t) + (k_1 + k_2)y_1(t) - c_2\dot{y}_2(t) - k_2y_2(t) = u_1(t)$$

$$m_2\ddot{y}_2(t) + c_2\dot{y}_2(t) + k_2y_2(t) - c_2\dot{y}_1(t) - k_2y_1(t) = u_2(t)$$

State-space Method to TF

- $\dot{\mathrm{x}} = A\mathrm{x} + B\mathrm{u}$
- $\mathrm{y} = C\mathrm{x} + D\mathrm{u}$

$\rightarrow$ $sX(s) - x(0) = AX(s) + BU(s)$: taking Laplace Transform

$\rightarrow$ $Y(s) = CX(s) + DU(s)$: taking Laplace Transform

$\Rightarrow$ $sX(s) = AX(s) + BU(s)$: $x(0) = 0$

$\rightarrow$ $(sI - A)X(s) = BU(s)$

$\Rightarrow$ $X(s) = (sI - A)^{-1}BU(s)$

$\Rightarrow$ $Y(s) = \left[C(sI - A)^{-1}B + D\right] U(s)$

$$G(s) = C(sI - A)^{-1}B + D \qquad : x(0) = 0$$

Example: Mechanical System

- State Equation:
$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\frac{k}{m} & -\frac{b}{m}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{m}\end{bmatrix} u \Rightarrow \dot{x} = Ax(t) + Bu(t)$$
- Output Equation: $y = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} \Rightarrow y = Cx(t) + Du(t)$

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D$$
$$= [1 \quad 0]\left\{\begin{bmatrix}s & 0\\ 0 & s\end{bmatrix} - \begin{bmatrix}0 & 1\\ -\frac{k}{m} & -\frac{b}{m}\end{bmatrix}\right\}^{-1}\begin{bmatrix}0\\ \frac{1}{m}\end{bmatrix} + 0$$
$$= [1 \quad 0]\begin{bmatrix}s & -1\\ \frac{k}{m} & s + \frac{b}{m}\end{bmatrix}^{-1}\begin{bmatrix}0\\ \frac{1}{m}\end{bmatrix}$$

T2104

$$G(s) = [1 \quad 0]\frac{1}{s^2 + \frac{b}{m}s + \frac{k}{m}}\begin{bmatrix}s + \frac{b}{m} & 1\\ -\frac{k}{m} & s\end{bmatrix}\begin{bmatrix}0\\ \frac{1}{m}\end{bmatrix}$$
$$= \frac{1}{ms^2 + bs + k}$$

T2105

$$\begin{bmatrix}a & b\\ c & d\end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$$

Example: ▷ Obtain TF from State-space equations:

$$\mathbf{A} = \begin{bmatrix}-1 & 1 & 0\\ 0 & -1 & 1\\ 0 & 0 & -2\end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix}0\\ 0\\ 1\end{bmatrix}, \quad \mathbf{C} = [1 \quad 0 \quad 0], \quad D = 0$$

T2107

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}-1 & 1 & 0\\ 0 & -1 & 1\\ 0 & 0 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 1\end{bmatrix}u$$
$$y = [1 \quad 0 \quad 0]\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

T2106

$$G(s) = C(sI - A)^{-1}B + D$$

$$G(s) = [1 \quad 0 \quad 0]\begin{bmatrix}s + 1 & -1 & 0\\ 0 & s + 1 & -1\\ 0 & 0 & s + 2\end{bmatrix}^{-1}\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}$$
$$= [1 \quad 0 \quad 0]\begin{bmatrix}\frac{1}{s+1} & \frac{1}{(s+1)^2} & \frac{1}{(s+1)^2(s+2)}\\ 0 & \frac{1}{s+1} & \frac{1}{(s+1)(s+2)}\\ 0 & 0 & \frac{1}{s+2}\end{bmatrix}\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}$$
$$= \frac{1}{(s + 1)^2(s + 2)} = \frac{1}{s^3 + 4s^2 + 5s + 2}$$

T2108

Example: ▷ Obtain TF from State-space equations:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-1 & -1\\ 6.5 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}$$

$$\begin{bmatrix}y_1\\ y_2\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}$$

T2109

$Y(s) = G(s)U(s)$

$$\begin{aligned}\mathbf{G}(s) &= \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\\ &= \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}s+1 & 1\\ -6.5 & s\end{bmatrix}^{-1}\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}\\ &= \frac{1}{s^2+s+6.5}\begin{bmatrix}s & -1\\ 6.5 & s+1\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}\\ &= \frac{1}{s^2+s+6.5}\begin{bmatrix}s-1 & s\\ s+7.5 & 6.5\end{bmatrix}\end{aligned}$$

$$\begin{bmatrix}Y_1(s)\\ Y_2(s)\end{bmatrix} = \begin{bmatrix}\frac{s-1}{s^2+s+6.5} & \frac{s}{s^2+s+6.5}\\ \frac{s+7.5}{s^2+s+6.5} & \frac{6.5}{s^2+s+6.5}\end{bmatrix}\begin{bmatrix}U_1(s)\\ U_2(s)\end{bmatrix}$$

T2110

Solution of Homogeneous State Equations

- $\dot{x}(t) = Ax(t)$

$\longrightarrow$ $sX(s) - x(0) = AX(s)$

$\longrightarrow$ $(sI - A)X(s) = x(0)$

$\Rightarrow$ $X(s) = (sI - A)^{-1}x(0)$

$\Rightarrow$ $x(t) = \mathcal{L}^{-1}[(sI - A)^{-1}]x(0) = \phi(t)x(0)$

- State-transition matrix, $\phi(t) \equiv \mathcal{L}^{-1}[(sI - A)^{-1}]$
- $(sI - A)^{-1} = \frac{I}{s} + \frac{A}{s^2} + \frac{A^2}{s^3} + \cdots$
- $\phi(t) = \mathcal{L}^{-1}[(sI - A)^{-1}] = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots = e^{At}$

$$\boxed{x(t) = \phi(t)x(0) = e^{At}x(0)}$$

Example: ▷ Obtain state-transition matrix of the following system:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

- $\phi(t) = e^{At} = \mathcal{L}^{-1}[(sI - A)^{-1}]$
- $(sI - A) = \begin{bmatrix}s & 0\\ 0 & s\end{bmatrix} - \begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix} = \begin{bmatrix}s & -1\\ 2 & s+3\end{bmatrix}$

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{(s+1)(s+2)}\begin{bmatrix}s+3 & 1\\ -2 & s\end{bmatrix}$$

$$= \begin{bmatrix}\frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)}\\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)}\end{bmatrix}$$

T2111

$$\mathbf{\Phi}(t) = e^{\mathbf{A}t} = \mathcal{L}^{-1}[(s\mathbf{I} - \mathbf{A})^{-1}]$$

$$= \begin{bmatrix}2e^{-t} - e^{-2t} & e^{-t} - e^{-2t}\\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t}\end{bmatrix}$$

T2112

Solution of Non-Homogeneous State Equations

- $\dot{x}(t) = Ax(t) + Bu(t)$

→ $sX(s) - x(0) = AX(s) + BU(s)$

→ $(sI - A)X(s) = x(0) + BU(s)$

⇒ $X(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}BU(s)$

⇒ $X(s) = \mathcal{L}[e^{At}]x(0) + \mathcal{L}[e^{At}]BU(s)$

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \qquad : t \geq 0$$

- $e^{At}x(0) \equiv$ free response: depends on initial condition, $x(0)$, if initial condition is zero this term is also zero.
- $\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \equiv$ forced response: depends on input $u(t) \in [0, t]$, if no forced input this term is zero.
  - If initial time is given by t_0 instead of 0:

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

Example: ▷ Obtain state-transition matrix of the following system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$\mathbf{\Phi}(t) = e^{\mathbf{A}t} = \mathscr{L}^{-1}\left[(s\mathbf{I} - \mathbf{A})^{-1}\right]$$

T2112

$$= \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

The response to the unit-step input is then obtained as:

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t \begin{bmatrix} 2e^{-(t-\tau)} - e^{-2(t-\tau)} & e^{-(t-\tau)} - e^{-2(t-\tau)} \\ -2e^{-(t-\tau)} + 2e^{-2(t-\tau)} & -e^{-(t-\tau)} + 2e^{-2(t-\tau)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} [1]\, d\tau$$

T2113

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

If the initial state is zero, or x(0)=0, then x(t) can be simplified to:

T2114

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

Example: ▷ $\ddot{y}(t) + 7\dot{y}(t) + 12y(t) = u(t)$,
$u(t)$ is a step input of magnitude 3, and $y(0) = 0.10$ and $\dot{y}(0) = 0.05$.

- $$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -12 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$
- $(sI - A)^{-1} = \frac{1}{s^2+7s+12} \begin{bmatrix} 0 & 1 \\ -12 & -7 \end{bmatrix}$
- $$X(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}BU(s)$$
  $$= \frac{1}{s^2 + 7s + 12} \begin{bmatrix} s+7 & 1 \\ -12 & s \end{bmatrix} \begin{bmatrix} 0.10 \\ 0.05 \end{bmatrix} + \frac{1}{s^2 + 7s + 12} \begin{bmatrix} s+7 & 1 \\ -12 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{3}{s}$$
  $$= \frac{1}{(s+3)(s+4)} \begin{bmatrix} 0.10s + 075 + \frac{3}{s} \\ 0.05s + 1.80 \end{bmatrix}$$
- $X_1(s) = \frac{0.10s^2+0.75s+3}{s(s+3)(s+4)} = \frac{0.25}{s} - \frac{0.55}{s+3} + \frac{0.40}{s+4}$

⇒ $y(t) = x_1(t) = \mathcal{L}^{-1}[X_1(s)] = 0.25 - 0.55e^{-3t} + 0.40e^{-4t} : t \geq 0$

Stability in State Space

- $\dot{\mathrm{x}} = A\mathrm{x} + B\mathrm{u} \rightarrow X(s) = (sI - A)^{-1}BU(s)$
- $\mathrm{y} = C\mathrm{x} + D\mathrm{u} \rightarrow Y(s) = \left[C(sI - A)^{-1}B + D\right] U(s)$

$\rightarrow$ $(sI - A)^{-1} = \frac{\mathrm{adj}(sI-A)}{\det(sI-A)}$

- $\det(sI - A) = 0$ is the characteristic equation from which the system poles can be found.
- $\det(sI - A) = 0 \Rightarrow P(s) = 0$.
- Use RH stability criteria using $P(s) = 0$.

Example: ▷ Investigate the stability:

$$\dot{x} = \begin{bmatrix} 0 & 3 & 1 \\ 2 & 8 & 1 \\ -10 & -5 & -2 \end{bmatrix} x + \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} u; \quad y = [1 \quad 0 \quad 0]\, x$$

$$(s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 3 & 1 \\ 2 & 8 & 1 \\ -10 & -5 & -2 \end{bmatrix} = \begin{bmatrix} s & -3 & -1 \\ -2 & s-8 & -1 \\ 10 & 5 & s+2 \end{bmatrix}$$

T2121

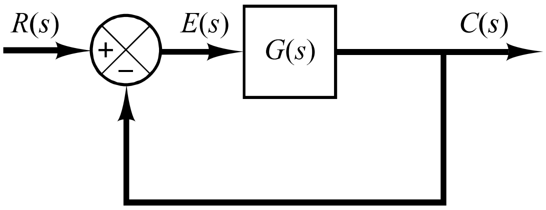
$$\det(sI - A) = s^3 - 6s^2 - 7s - 52 = 0 = P(s)$$

s^3	1	−7
s^2	~~−6~~ − 3	~~−52~~ − 26
s^1	~~$-\frac{47}{3}$~~ − 1	~~0~~ 0
s^0	−26	

T2122

One sign change in the first column, so the system is unstable.

Steady-State Error for Systems in State -Space



T2054

- $e(t) = r(t) - c(t) \ \rightarrow E(s) = R(s) - C(s) \Rightarrow E(s) = R(s)\left[1 - G(s)\right]$
- $\dot{\mathrm{x}} = A\mathrm{x} + B\mathrm{u}; \quad \mathrm{y} = C\mathrm{x} + D\mathrm{u} \Rightarrow G(s) = C(sI - A)^{-1}B + D$
- Applying final-value theorem:

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} sR(s)\left[1 - \left\{C(sI - A)^{-1}B + D\right\}\right]$$

- For unit step input, $R(s) = 1/s$, and for unit ramp input, $R(s)1/s^2$.

Example: ▷ Estimate steady-state error for a) unit step input b) unit ramp input:

$$A = \begin{bmatrix} -5 & 1 & 0 \\ 0 & -2 & 1 \\ 20 & -10 & 1 \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}; \quad C = \begin{bmatrix} -1 & 1 & 0 \end{bmatrix}$$

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} sR(s)\left[1 - \left\{C(sI - A)^{-1}B + D\right\}\right]$$

$$e(\infty) = \lim_{s\to 0} sR(s)\left(1 - \frac{s+4}{s^3 + 6s^2 + 13s + 20}\right)$$

$$= \lim_{s\to 0} sR(s)\left(\frac{s^3 + 6s^2 + 12s + 16}{s^3 + 6s^2 + 13s + 20}\right)$$

T2123

a) $R(s) = 1/s \rightarrow e_{ss} = 4/5$

b) $R(s) = 1/s^2 \rightarrow e_{ss} = \infty$

Controllability

If an input to a system can be found that takes every state variable from a desired initial state to a desired final state, the system is said to be controllable; otherwise, the system is uncontrollable.

- An n^{th}-order plant whose state equation:

$$\dot{x} = Ax + Bu$$

- Completely controllable if the matrix:

$$P = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$

is of rank n, where P is called the controllability matrix.

Example: ▷ Investigate controllability.

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} u(t)$$

T2135

$$B = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 7 \end{bmatrix}$$

$$A^2B = A(AB) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ 7 \end{bmatrix} = \begin{bmatrix} -3 \\ 7 \\ -15 \end{bmatrix}$$

$$P = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & 1 & -3 \\ 1 & -3 & 7 \\ -3 & 7 & -15 \end{bmatrix}$$

T2116

- $|P| = 0$, rank of P is less than 3.
- Not-controllable.

Example: ▷ Investigate controllability.

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} b_0 & b_1 & b_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

T2117

$$B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad AB = \begin{bmatrix} 0 \\ 1 \\ -a_2 \end{bmatrix}$$

$$A^2B = \begin{bmatrix} 1 \\ -a_2 \\ a_2^2 - a_1 \end{bmatrix}$$

$$P = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -a_2 \\ 1 & -a_2 & a_2^2 - a_1 \end{bmatrix}$$

T2118

- $|P| = -1 \neq 0$.
- So, controllable.

Observability

If the initial-state vector, $x(t_0)$, can be found from $u(t)$ and $y(t)$ measured over a finite interval of time from t_0, the system is said to be observable; otherwise the system is said to be unobservable.

- An n^{th}-order plant whose state equation & output equation:

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

- Completely observable if the matrix:

$$Q = \begin{bmatrix} C \\ CA \\ CA^2 \\ \cdot \\ \cdot \\ CA^{n-1} \end{bmatrix}$$

is of rank n, where Q is called the observability matrix.

Example: ▷ Investigate observability.

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

$$y(t) = [\,0 \quad 1 \quad -3\,] \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

T2119

$$C = [\,0 \quad 1 \quad -3\,]$$

$$CA = [\,0 \quad 1 \quad -3\,] \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix} = [\,1 \quad -3 \quad 7\,]$$

$$CA^2 = (CA)A = [\,1 \quad -3 \quad 7\,] \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix} = [\,-3 \quad 7 \quad -15\,]$$

$$Q = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -6 \\ 1 & -3 & 7 \\ -3 & 7 & -15 \end{bmatrix}$$

T2120

- $|Q| = 0$, thus rank of Q is less than 3.
- So, non-observable.