Transient Heat Conduction

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ME 307: Heat Transfer Equipment Design
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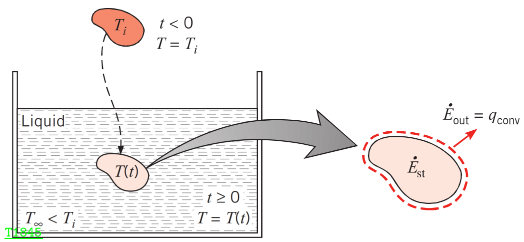
The Lumped Capacitance Method



$$-\dot{E}_{\text{out}} = \dot{E}_{\text{st}}$$

$$\dot{E}_{\text{out}} = hA(T - T_\infty)$$

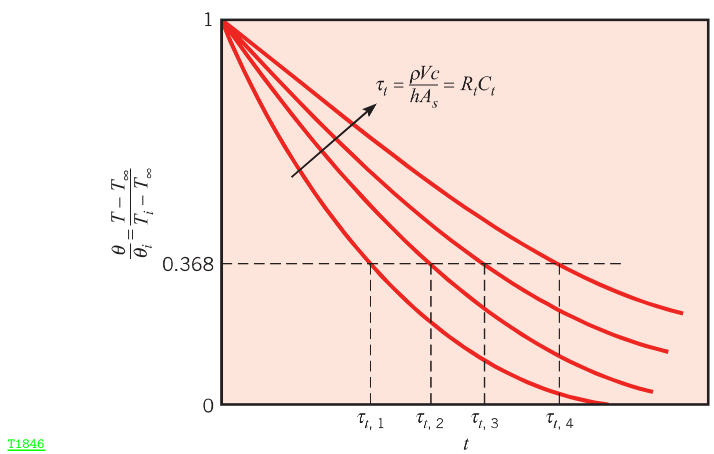
$$\dot{E}_{\text{st}} = \rho V c \frac{dT}{dt}$$

$$\theta = T - T_\infty$$

$$\frac{\rho V c}{hA}\frac{d\theta}{dt} = -\theta$$

$$\frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty} = \exp\left[-\frac{hA}{\rho V c}t\right] = \exp\left[-\frac{t}{\tau_t}\right]$$

- Thermal time constant, $\tau_t = (\frac{1}{hA})(\rho V c) = R_t C_t$
- R_t is the resistance to convection heat transfer,
- C_t is the lumped thermal capacitance of the solid.



Transient temperature response of lumped capacitance solids for different thermal time constants τ_t.

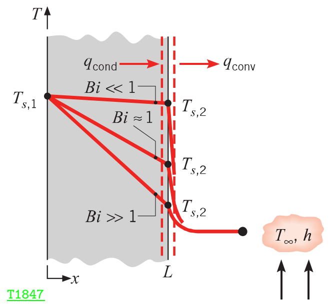
- Characteristic length, $L_c = \frac{V}{A}$.

$$\frac{hA}{\rho V c}t = \frac{ht}{\rho c L_c} = \frac{hL_c}{k}\frac{k}{\rho c}\frac{t}{L_c^2} = \frac{hL_c}{k}\frac{\alpha t}{L_c^2}$$

- Biot number, $B_i \equiv \frac{hL_c}{k} = \frac{(L/kA)}{(1/hA)} = \frac{R_{t,cond}}{R_{t,conv}}$, is the ratio of a conduction thermal resistance to a convection resistance. The Biot number approaches zero when the conductivity of the solid or the convection resistance is so large that the solid is practically isothermal and the temperature change is mostly in the fluid at the interface.
- Fourier number, $F_o \equiv \frac{\alpha t}{L_c^2} = \frac{Ak/L_c}{(\rho c V)/t}$, the ratio of the rate of heat transfer by conduction to the rate of energy storage in the system.

$$\frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty} = \exp\left[-\frac{t}{\tau_t}\right] = \exp(-B_i \cdot F_o)$$

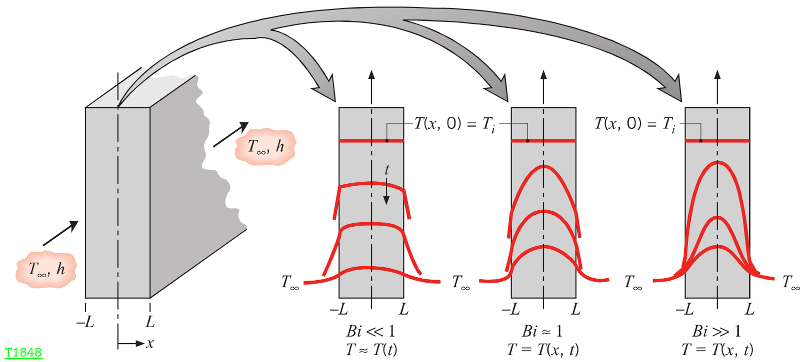
Validity of the Lumped Capacitance Method



$$\frac{kA}{L}(T_{s,1} - T_{s,2}) = hA(T_{s,2} - T_\infty)$$

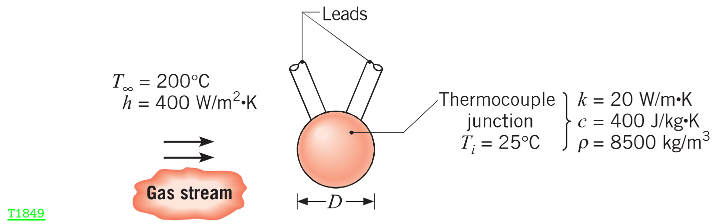
$$\frac{T_{s,1} - T_{s,2}}{T_{s,2} - T_\infty} = \frac{(L/kA)}{(1/hA)} = \frac{R_{t,cond}}{R_{t,conv}} = \frac{hL}{k} \equiv B_i$$

If Biot number, $B_i << 1$, resistance to conduction within the solid is much less than resistance to convection across fluid boundary layer. Hence, uniform temperature distribution within the solid is reasonable if the Biot number is small.



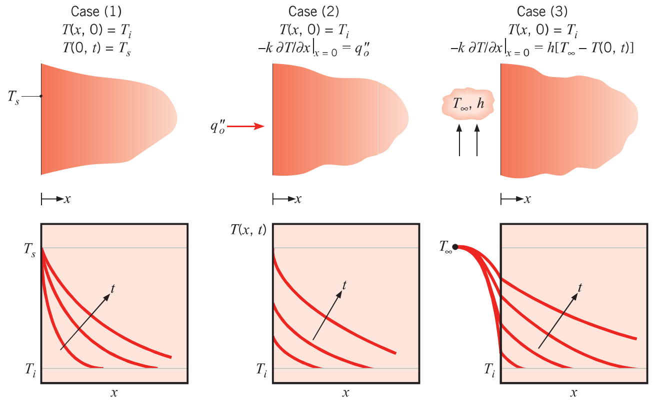
Transient temperature distributions for different Biot numbers in a plane wall symmetrically cooled by convection.

Example: ▷ A thermocouple junction, which may be approximated as a sphere, is to be used for temperature measurement in a gas stream. Determine the junction diameter needed for the thermocouple to have a time constant of 1 s. If the junction is at 25°C and is placed in a gas stream that is at 200°C, how long will it take for the junction to reach 199°C?



[0.706 mm, 5.17 s]

Semi-infinite Solid



T1850

Transient temperature distributions in a semi-infinite solid for three surface conditions: constant surface temperature, constant surface heat flux, and surface convection.

Case 1 Constant Surface Temperature: $T(0, t) = T_s$

$$\frac{T(x,t) - T_s}{T_i - T_s} = \text{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

$$q''_s(t) = \frac{k(T_s - T_i)}{\sqrt{\pi \alpha t}}$$

Case 2 Constant Surface Heat Flux: $q''_s = q''_o$

T1853

$$T(x,t) - T_i = \frac{2q''_o(\alpha t/\pi)^{1/2}}{k}\exp\left(\frac{-x^2}{4\alpha t}\right) - \frac{q''_o x}{k}\text{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

Case 3 Surface Convection: $-k\left.\frac{\partial T}{\partial x}\right|_{x=0} = h[T_\infty - T(0,t)]$

$$\frac{T(x,t) - T_i}{T_\infty - T_i} = \text{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) - \left[\exp\left(\frac{hx}{k} + \frac{h^2\alpha t}{k^2}\right)\right]\left[\text{erfc}\left(\frac{x}{2\sqrt{\alpha t}} + \frac{h\sqrt{\alpha t}}{k}\right)\right]$$

T1854

Gaussian Error Function

w	erf w	w	erf w	w	erf w
0.00	0.00000	0.36	0.38933	1.04	0.85865
0.02	0.02256	0.38	0.40901	1.08	0.87333
0.04	0.04511	0.40	0.42839	1.12	0.88679
0.06	0.06762	0.44	0.46622	1.16	0.89910
0.08	0.09008	0.48	0.50275	1.20	0.91031
0.10	0.11246	0.52	0.53790	1.30	0.93401
0.12	0.13476	0.56	0.57162	1.40	0.95228
0.14	0.15695	0.60	0.60386	1.50	0.96611
0.16	0.17901	0.64	0.63459	1.60	0.97635
0.18	0.20094	0.68	0.66378	1.70	0.98379
0.20	0.22270	0.72	0.69143	1.80	0.98909
0.22	0.24430	0.76	0.71754	1.90	0.99279
0.24	0.26570	0.80	0.74210	2.00	0.99532
0.26	0.28690	0.84	0.76514	2.20	0.99814
0.28	0.30788	0.88	0.78669	2.40	0.99931
0.30	0.32863	0.92	0.80677	2.60	0.99976
0.32	0.34913	0.96	0.82542	2.80	0.99992
0.34	0.36936	1.00	0.84270	3.00	0.99998

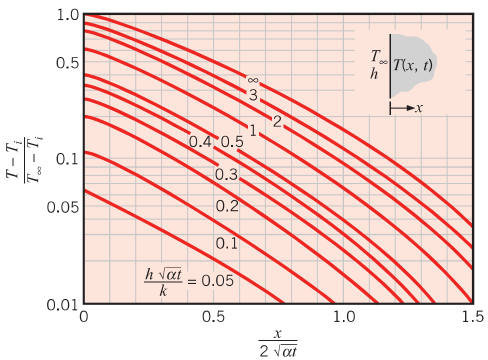
[1]The Gaussian error function is defined as

$$\text{erf } w = \frac{2}{\sqrt{\pi}}\int_0^w e^{-v^2}\,dv$$

The complementary error function is defined as

$$\text{erfc } w \equiv 1 - \text{erf } w$$

T1852



T1851

Transient temperatures for a semi-infinite solid with surface heat transfer.

- For $h = \infty$, surface instantaneously achieves the imposed fluid temperature ($T_s = T_\infty$), and with the second term on the right-hand side of Case 3 to zero, the result is equivalent to Case 1.

Example: ▷ A large block of steel [k = 45 W/m °C, $\alpha = 1.4 \times 10^{-5} m^2/s$] is initially at a uniform temperature of 35°C. The surface is exposed to a heat flux (a) by suddenly raising the surface temperature to 250°C and (b) through a constant surface heat flux of 3.2×10^5 W/m^2. Calculate the temperature at a depth of 2.5 cm after a time of 0.5 min for both these cases.

[118.5°C, 79.3°C]

Example: ▷ A large aluminium slab [k = 215 W/m °C, $\alpha = 8.4 \times 10^{-5} m^2/s$] at a uniform temperature of 200°C suddenly has its surface temperature lowered to 70°C. What is the total heat removed from the slab per unit surface area when the temperature at a depth 4.0 cm has dropped to 120°C?

[21.14 MJ/m^2]

Example: ▷ Estimate the minimum depth at which one must place a water main below the surface to avoid freezing. The soil is initially at a uniform temperature of 20°C. Assume that under the worst conditions anticipated it is subjected to a surface temperature of -15°C for a period of 60 days.
Use the following properties for soil (300 K): $\rho = 2050$ kg/m^3, $k = 0.52$ W/mK, $c = 1840$ J/kgK, $\alpha = 0.138$ x 10^{-6} m^2 s.

[0.677 m]