System Dynamics

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ME 475: Mechatronics

Basic System Models

- **Modeling** is the process of representing the behavior of a system by a collection of mathematical equations & logics. Modeling is comprehensively utilized to study the response of any system. **Response** of a system is a measure of its fidelity to its purpose.
- **Simulation** is the process of solving the model and it performed on a computer.
- **Equations** are used to describe the relationship between the input and output of a system.

$$\textit{Input} \Longrightarrow \boxed{\text{Governing Equations}} \Longrightarrow \textit{Output}$$

- **Analogy** approach is also widely used to study system response.

Modeling of a General System

A dynamic system can be represented in general by the differential equation:

$$a_n \frac{d^n x}{dt^n} + a_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + \cdots + \underbrace{a_2 \frac{d^2x}{dt^2} + \underbrace{a_1 \frac{dx}{dt} + \overbrace{a_0 x = F(t)}^{0^{th}\ order}}_{1^{st}\ order}}_{2^{nd}\ order}$$

$F(t)$ $\equiv$ Input or the forcing function,
$x(t)$ $\equiv$ Output or the response of the system,
a's $\equiv$ Constants, physical system parameters

$\hookrightarrow$ **Order of a system is designated by the order of the D.E.**

Zeroth Order System

$$\boxed{x = \Bbbk\, F(t)}$$

- $\Bbbk \equiv \frac{1}{a_o} \Longleftrightarrow$ Static sensitivity or gain. It represents scaling between the input and the output.
- No equilibrium seeking force is present.
- Output follows the input without distortion or time lag.
- System requires no additional dynamic considerations.
- Represents ideal dynamic performance.
- `Example:` Potentiometer, ideal spring etc.

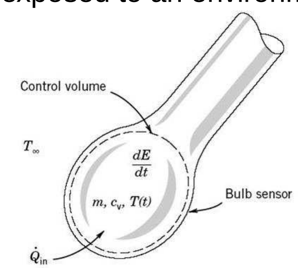
First Order System

$$\boxed{\tau \frac{dx}{dt} + x = \mathbb{k}\, F(t)}$$

$\tau = [a_1/a_o] \iff$ time constant of the system,
$a_o \iff$ dissipation (electric or thermal resistance).
$a_1 \iff$ storage (electric or thermal capacitance).

$\looparrowright$ Example: Thermometer, capacitor etc.

▶ Consider a thermocouple initially at temperature, T is suddenly exposed to an environment at T_∞.



$$\rightsquigarrow \dot{Q}_{in} = hA(T_\infty - T) = mC_v \frac{dT}{dt}$$

$$\Longrightarrow \tau \frac{dT}{dt} + T = T_\infty; \qquad \tau = \frac{mC_v}{hA}$$

$$m \uparrow C_v \uparrow \; h \downarrow A \downarrow \Longrightarrow \tau \uparrow$$

Response of a 1^{st} Order System: Step Input

$$\boxed{x = x_o,\; F = 0 : t = 0; \qquad F(t) = A : t > 0}$$

$$\tau \frac{dx}{dt} + x = \mathbb{k}\, F(t)$$

$$\Longrightarrow x(t) = A\mathbb{k}\left[1 - \exp^{-t/\tau}\right] + x_o e^{-t/\tau}$$

▶ $x(t \to \infty) = A\mathbb{k} = x_\infty \Longleftarrow$ Steady State Response
▶ Error, $e_m = x_\infty - x(t) = (x_\infty - x_o)e^{-t/\tau}$
▶ Nondimensional Error, $e_m/(x_\infty - x_o) = e^{-t/\tau}$

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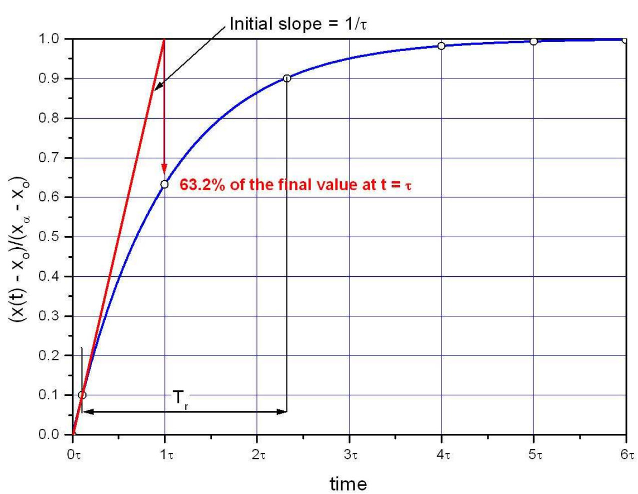

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- **Time Constant,** τ - time required to complete 63.2% of the process.
- **Rise Time,** T_r - time required to achieve response from 10% to 90% of final value.
  $\looparrowright$ For first order system, $T_r = 2.31\tau - 0.11\tau = 2.2\tau$.
- **Settling Time,** T_s - the time for the response to reach, and stay within 2% of its final value.
  $\looparrowright$ For first order system, $T_s = 4\tau$.
- Process is assumed to be completed when $t \geq 5\tau$.
- Faster response is associated with shorter τ.

Transfer Function (TF)

- Transfer function of a system, $G(s)$, is defined as the ratio of the Laplace Transform (LT) of the output variable, $X(s)$, to the LT of the input variable, $F(s)$, *with all the initial conditions are assumed to be zero.*

$$G(s) = \frac{X(s)}{F(s)}$$

  $s \equiv \sigma + j\omega$ is the complex variable of the L.T.
- Amplitude Ratio, $G_a = |G(j\omega)|$
- Phase Angle, $\phi = \angle G(j\omega)$
- Time Delay, $\Delta t = \frac{\phi}{\omega}$

$$\Longrightarrow \boxed{x(t) = G_a\, F(t)\, \angle\phi} \Longleftarrow$$

TF of a 1st Order System

$$\tau \frac{dx}{dt} + x = \Bbbk\, F(t)$$

- $\frac{d^n x}{dt^n} \Longrightarrow s^n X(s), \qquad F(t) \Longrightarrow F(s).$

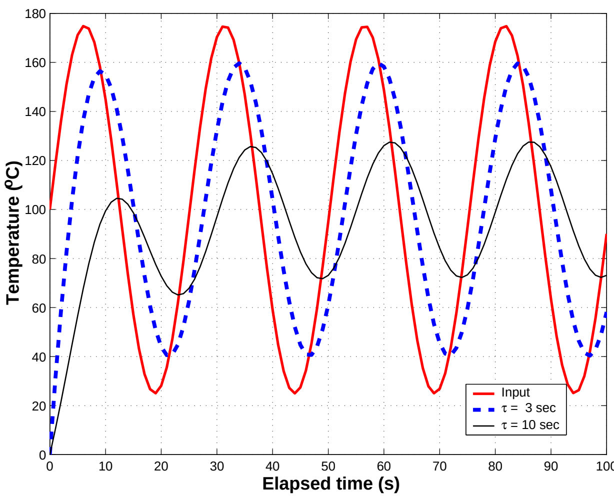
$$\Rightarrow \tau s X(s) + X(s) = \Bbbk\, F(s)$$

$$F(s) \Longrightarrow \boxed{\frac{\Bbbk}{\tau s + 1}} \Longrightarrow X(s)$$

- $s \Longleftarrow j\omega$
- $G_a = |G(j\omega)| = \left|\frac{\Bbbk}{j\omega\tau + 1}\right| = \frac{\Bbbk}{\sqrt{1+(\omega\tau)^2}}$
- $\phi = \angle G(j\omega) = \tan^{-1}(-\omega\tau)$

Response of a 1st Order System: Harmonic Input

Two thermometers ($\tau_1 = 3$s & $\tau_2 = 10$s), initially at 0°C are exposed to $T(^oC) = 100 + 75\sin(0.25t)$



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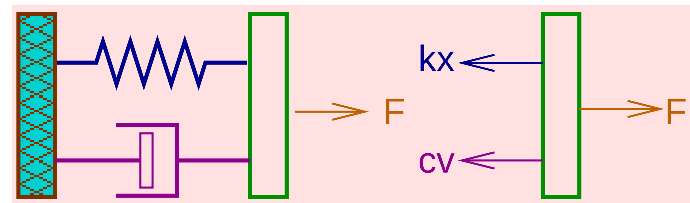
Item	τ	ϕ	Δt	G_a
01	3 sec	-36.6^o	-2.54 sec	0.80
02	10 sec	-68.2^o	-4.76 sec	0.37

- Response to harmonic input is
  - at same frequency,
  - with a phase shift (time lag), and
  - reduced amplitude.
- The larger the time constant, the greater the phase lag & amplitude decrease (attenuation).

. . . Matlab Codes

```
% Example 01
num  = [1.0];  den01 = [ 3.0 1.0];  den02 = [10.0 1.0];
sys01 = tf(num, den01); sys02 = tf(num, den02);
time = [0:1:100];
T_inp = 100.0+75*sin(0.25*time);
[T_out01,time] = lsim(sys01,T_inp,time);
[T_out02,time] = lsim(sys02,T_inp,time);
plot(time,T_inp, 'r-','LineWidth', 2.0); hold on;
plot(time,T_out01, 'b-.','LineWidth', 4.0); hold on;
plot(time,T_out02, 'k--','LineWidth', 4.0); hold off;
xlabel('\bf{\fontsize{14} Elapsed time (s)}')
ylabel('\bf{\fontsize{14} Temperature (^oC)}')
grid on
legend ('\bf{\fontsize{12}{Input', '\tau =  3 s', '\tau = 10 s}}',4)
title ('\bf{\fontsize{14}Response of a 1^{st} Order System for a Hamonic Input}')
```


Second Order System



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$$F - kx - c\frac{dx}{dt} = m\frac{d^2x}{dt^2} \Longrightarrow m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F$$

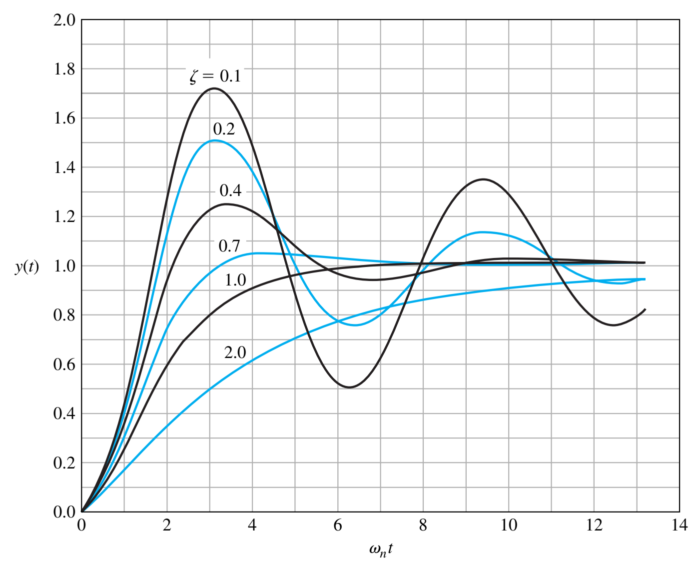
ω_n	$\equiv$	$\sqrt{\frac{k}{m}}$	$\Longleftrightarrow$	undamped natural frequency (rad/s)
c_c	$\equiv$	$2\sqrt{mk}$	$\Longleftrightarrow$	critical damping coefficient
ζ	$\equiv$	c/c_c	$\Longleftrightarrow$	damping ratio

TF of a 2^{nd} Order System

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t) \Longrightarrow \frac{1}{\omega_n^2}\frac{d^2x}{dt^2} + 2\frac{\zeta}{\omega_n}\frac{dx}{dt} + x = \frac{F(t)}{k}$$

- $G(s) = \frac{1/k}{\frac{1}{\omega_n^2}s^2 + 2\frac{\zeta}{\omega_n}s + 1} = \frac{\omega_n^2/k}{s^2 + 2\zeta\omega_n s + \omega_n^2}$
- $G(j\omega) = \frac{1/k}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right] + j\left[2\zeta\frac{\omega}{\omega_n}\right]}$
- $G_a = |G(j\omega)| = \frac{1/k}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + 4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2}}$
- $\phi = \angle G(j\omega) = \tan^{-1}\left[-\frac{2\zeta\frac{\omega}{\omega_n}}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]}\right]$

Response of a 2^{nd} Order System: Step Input



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- Steady state position is obtained after a long period of time.
- Underdamped system ($\zeta < 1$): response overshoots the steady-state value initially, & then eventually decays to the steady-state value. The smaller the value of ζ, the larger the overshoot.
- Critical damping ($\zeta = 1$): an exponential rise occurs to approach the steady-state value without any overshoot.
- Overdamped ($\zeta > 1$): the system approaches the steady-state value without overshoot, but at a slower rate.

. . . Matlab Codes

```
% Example 02
time = [0.0:0.1:20]; num = [1];
zeta01 = 0.0001; den01 = [1 2*zeta01 1]; sys01 = tf(num,den01);
zeta02 = 0.3;    den02 = [1 2*zeta02 1]; sys02 = tf(num,den02);
zeta03 = 0.7;    den03 = [1 2*zeta03 1]; sys03 = tf(num,den03);
zeta04 = 1.0;    den04 = [1 2*zeta04 1]; sys04 = tf(num,den04);
zeta05 = 4.0;    den05 = [1 2*zeta05 1]; sys05 = tf(num,den05);
[Y0]  = 1.0+0.0*time;
[X01,T1] = step(sys01,time); [X02,T2] = step(sys02,time);
[X03,T3] = step(sys03,time); [X04,T4] = step(sys04,time);
[X05,T5] = step(sys05,time);
plot(time, Y0,'r-','LineWidth',3.0); hold on;
plot(T1, X01,'b-.','LineWidth',1.5); hold on;
plot(T2, X02,'k-.','LineWidth',2.5); hold on;
plot(T3, X03,'b-','LineWidth',3.0); hold on;
plot(T4, X04,'k--','LineWidth',4.0); hold on;
plot(T5, X05,'r-.','LineWidth',1.0); hold on;
xlabel('\bf{\fontsize{14} \omega_n t }')
ylabel('\bf{\fontsize{14} X(t)}')
title ('\bf{\fontsize{14}Response of a 2^{nd} Order System for Step Input}')
legend('\bf{\fontsize{10}{Input','\zeta = 0.0001','\zeta = 0.3','\zeta = 0.7','\zeta = 1.0','\zeta = 4.
```


Response of a 2^{nd} Order System: Harmonic Input

$$F(t) = \begin{cases} 0 & t = 0 \\ A \sin \omega t & t > 0 \end{cases}$$

- System has a good *linearity* for low damping ratios and up to a frequency ratio of 0.3 since the amplitude gain is very nearly unity ($G_a \simeq 1$).
- For large values of ζ, the amplitude is reduced substantially.
- The phase shift characteristics are a strong function of frequency ratio for all frequencies.
- As a general rule of thumb, the choice of $\zeta = 0.707$ **is optimal** since it results in the best combination of amplitude linearity and phase linearity over the widest range of frequencies.

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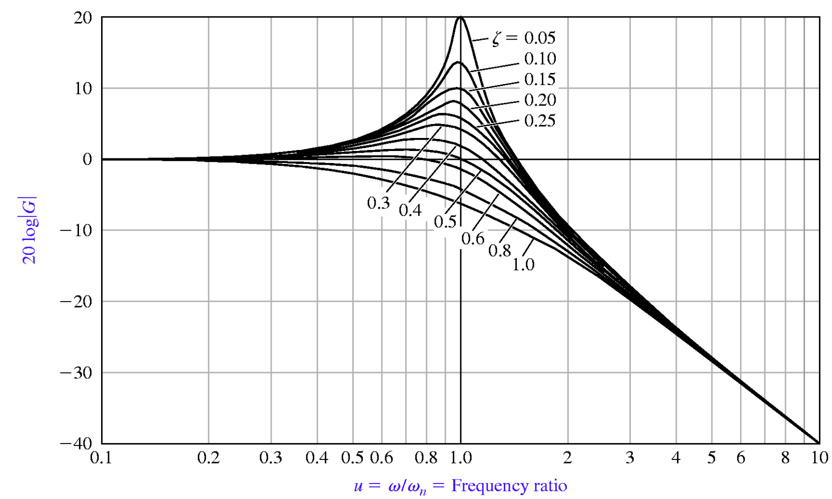

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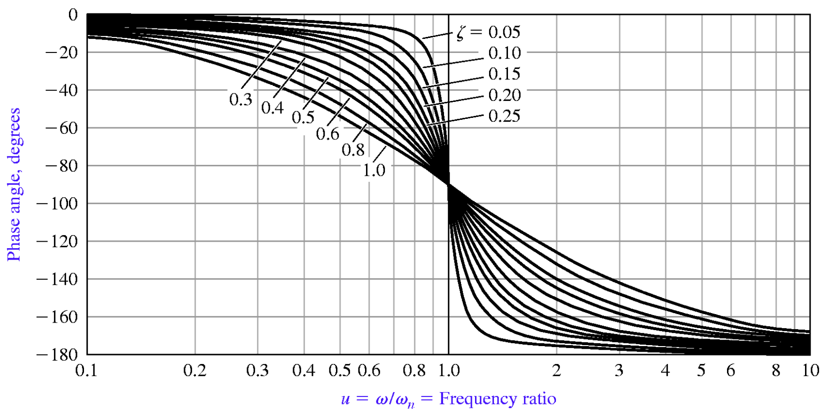

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. . . Matlab Codes

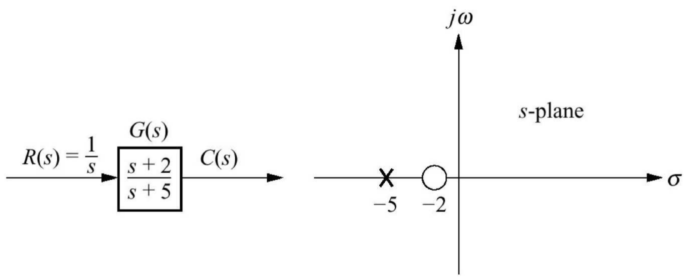
```
% Example 03
time = [0.1:0.05;10];
num = [1];
zeta01 = 0.0001; den01 = [1 2*zeta01 1]; sys01 = tf(num,den01);
zeta02 = 0.1;    den02 = [1 2*zeta02 1]; sys02 = tf(num,den02);
zeta03 = 0.7;    den03 = [1 2*zeta03 1]; sys03 = tf(num,den03);
zeta04 = 1.0;    den04 = [1 2*zeta04 1]; sys04 = tf(num,den04);
zeta05 = 4.0;    den05 = [1 2*zeta05 1]; sys05 = tf(num,den05);
grid on;
bode(sys01,sys02,sys03,sys04,sys05);
```


Poles & Zeros of a Transfer Function

- The concept of poles and zeros significantly simplifies the evaluation of a system's response.
- **Poles** of a transfer function are:
  1. the values of LT variable, s, that cause the TF to become infinite, or
  2. any roots of the denominator of the TF that are common to roots of the numerator.
- **Zeros** of a TF are:
  1. the values of LT variable, s, that cause the TF to become zero, or
  2. any roots of the numerator of the TF that are common to roots of the denominator.

- A pole of the input function generates the form of the *forced* response.
- A pole of the transfer function generates the form of the *natural* response.
- A pole on the real axis generates an *exponential* response of the form $e^{-\alpha t}$, where, $-\alpha$ is the pole location on the real axis. Thus, the further to the left a pole is on the negative real axis, the faster the exponential transient response will decay to zero.
- The zeros and poles generate the *amplitudes* for both the forced and natural responses.

Evolution of Response component: $1^{st}O$ System



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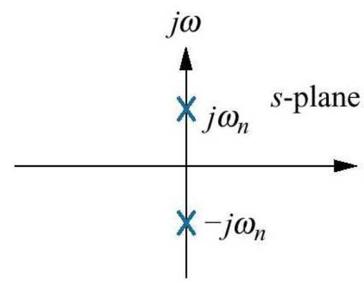
A pole exists at s = -5 and a zero exists at -2; these values are plotted on the complex s-plane with an $\times$ and a $\bigcirc$ for the pole and zero, respectively.

$$C(s) = R(s)G(s) = \frac{2/5}{s} + \frac{3/5}{s+5} \implies c(t) = \frac{2}{5} + \frac{3}{5}e^{-5t}$$

Undamped ($\zeta = 0$) of $2^{nd}O$ Systems

- Poles: Two imaginary at $\pm j\omega_1$
- Natural response: $c(t) = A\cos(\omega_1 t - \phi)$



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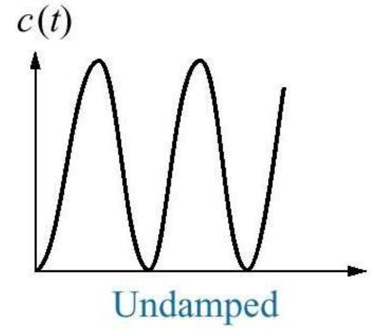

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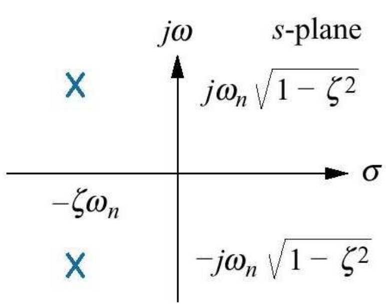
Under-damped ($0 < \zeta < 1$) of $2^{nd}O$ Systems

- Poles: Two complex at $-\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$
- Natural response: $c(t) = Ae^{-\zeta\omega_n t}\cos(\omega_n\sqrt{1-\zeta^2}t - \phi)$



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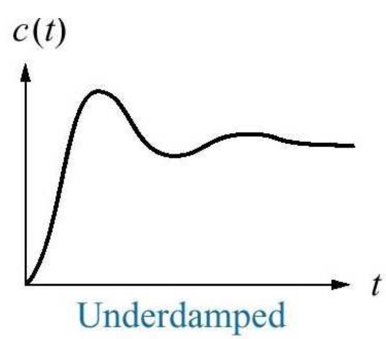

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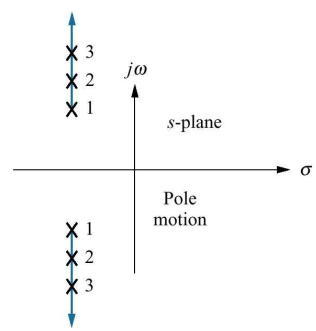

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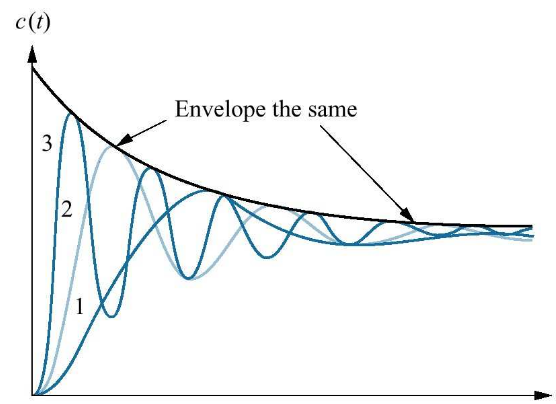

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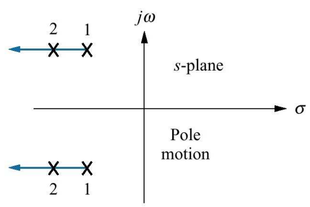
Step responses of second-order undamped systems as poles move with constant real part



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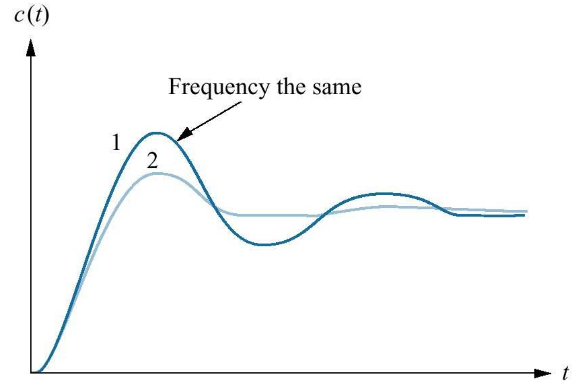

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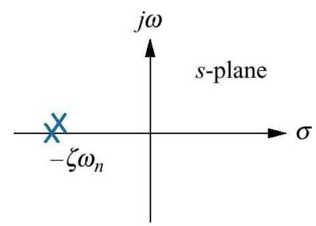
Step responses of second-order undamped systems as poles move with constant imaginary part

Critically-damped ($\zeta = 1$) of $2^{nd}O$ Systems

- Poles: Two real poles at $-\omega_n$
- Natural response: $c(t) = K_1 e^{-\omega_n} t + K_2 t e^{-\omega_n} t$



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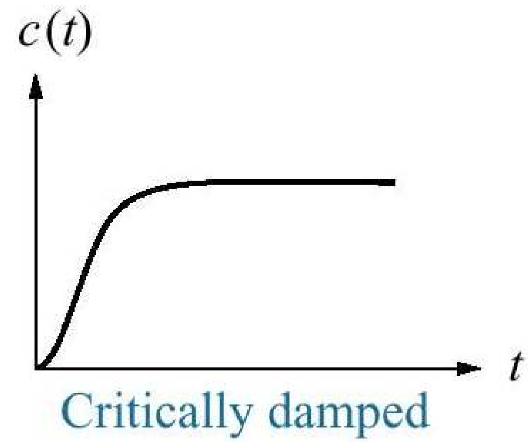
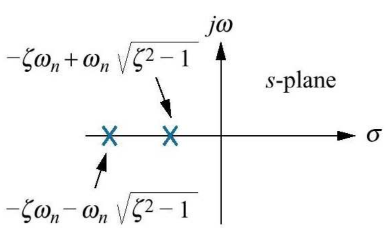


Over-damped ($\zeta > 1$) of $2^{nd}O$ Systems

- Poles: Two real poles at $-\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}, -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}$
- Natural response: $c(t) = K_1 e^{-\omega_n t} + K_2 t e^{-\omega_n t}$



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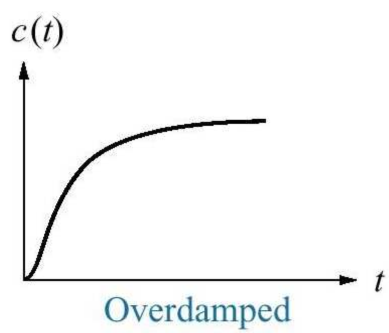

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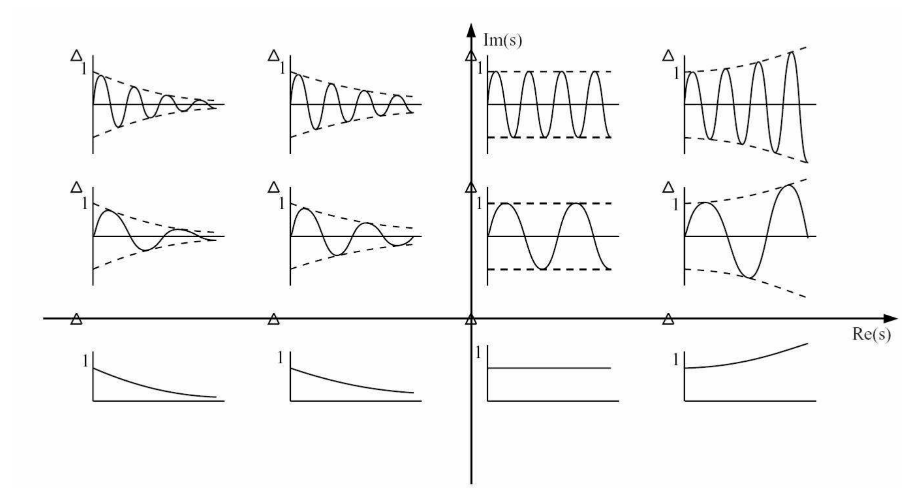
Impulse Responses of 2^{nd} Order Systems



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