Response of Measuring Systems, System Dynamics

Dr. Md. Zahurul Haq

Professor
Department of Mechanical Engineering
Bangladesh University of Engineering & Technology (BUET)
Dhaka-1000, Bangladesh
http://zahurul.buet.ac.bd/

ME 361: Instrumentation and Measuremen

http://zahurul.buet.ac.bd/ME361/

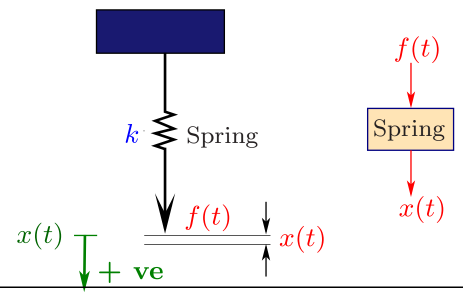
Basic System Models

- Modelling is the process of representing the behaviour of a system by a collection of mathematical equations & logics. It is comprehensively utilized to study the response of any system.
- Response of a system is a measure of its fidelity to its purpose.
- Simulation is the process of solving the model and it is performed using computer(s).
- Equations are used to describe the relationship between the input and output of a system.

$$\textit{Input} \Longrightarrow \boxed{\text{Governing Equations}} \Longrightarrow \textit{Output}$$

- Analogy approach is widely used to study system response.

Mechanical System Elements: (a) Spring

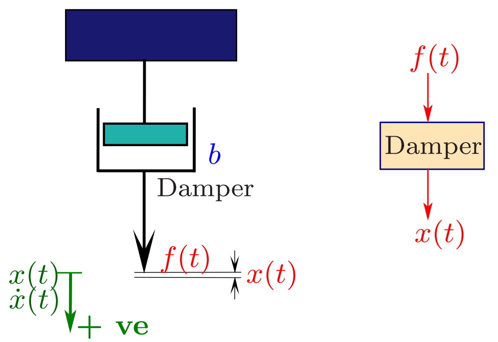


T848

$$f(t) = kx(t)$$

F	$\equiv$	Force (tension or compression),
x	$\equiv$	Displacement (extension or compression),
k	$\equiv$	Spring constant. The bigger the value of k the greater the forces required to stretch or compress the spring and so the greater the stiffness.

... (b) Dashpot/Damper



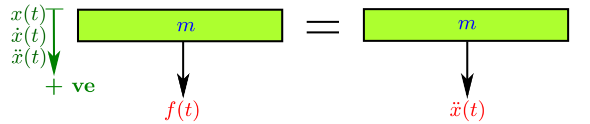
T849

$$f(t) = -bv = -b\frac{dx}{dt}$$

$F \equiv$ Force opposing the motion at velocity v,

$b \equiv$ Damping coefficient. Larger the value of b the greater the damping force at a particular velocity.

... (c) Mass



T850

$$f(t) = ma = m\frac{dv}{dt} = m\frac{d^2x}{dt^2}$$

$F \equiv$ Force required to cause acceleration, a,

$m \equiv$ Mass of the element that is distributed throughout some volume. However, in many cases, it is assumed to be concentrated at a point.

- Spring stores energy when stretched, and the energy is released when it springs back to its original state.

$$E = \frac{1}{2}\frac{f^2}{k}$$

- Energy is stored in mass when it is moving with a velocity, v, the energy being referred to as kinetic energy.

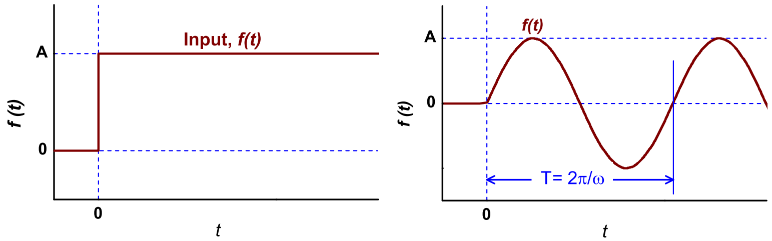
$$E = \frac{1}{2}mv^2$$

- Dashpot dissipates energy as heat rather than storing it, and dissipated power, P depends on the velocity, v.

$$P = bv^2$$

Step and Harmonic Input

- step function: $f(t) = \begin{cases} 0 & \text{at} \quad t \leq 0 \\ A & \text{for} \quad t > 0 \end{cases}$
- harmonic function: $f(t) = \begin{cases} 0 & \text{at} \quad t \leq 0 \\ A\sin\omega t & \text{for} \quad t > 0 \end{cases}$



T854 (a) Step input (b) Harmonic input

Step and harmonic inputs are widely used to analyse system response.

Modelling of a General Measurement System

The response of a measurement system, i.e., output, $x(t)$, when subjected to an input forcing function, $f(t)$, may be expressed by a linear ordinary differential equation with constant coefficients of the form:

$$a_n\frac{d^nx}{dt^n} + a_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + \cdots + \underbrace{a_2\frac{d^2x}{dt^2} + \underbrace{a_1\frac{dx}{dt} + \overbrace{a_0x = f(t)}^{0^{th}\ order}}_{1^{st}\ order}}_{2^{nd}\ order}$$

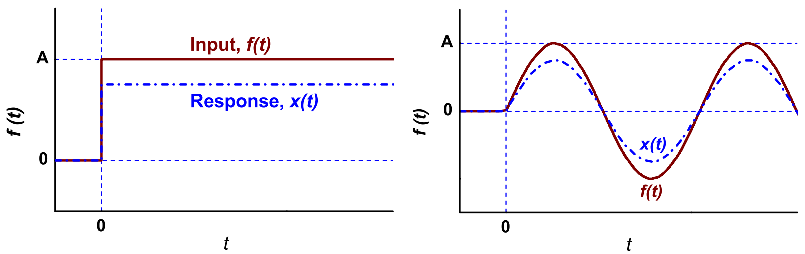
$f(t)$	$\equiv$	Input quantity imposed on the system,
$x(t)$	$\equiv$	Output or the response of the system,
a's	$\equiv$	Physical system parameters, assumed constants.

↬ Order of a system is designated by the order of the D.E.

Zeroth Order System

$$a_0x = f(t) :\Longrightarrow x(t) = \Bbbk\, f(t)$$

- $\Bbbk \equiv \frac{1}{a_o} \equiv$ Static sensitivity or gain: the scaling factor between the input and the output. For any-order system, it always has the same physical interpretation, i.e., the amount of output per unit input when the input is static and under such condition all the derivative terms of general equation are zero.
- No equilibrium seeking force is present.
- Output follows the input without distortion or time lag.
- System requires no additional dynamic considerations.
- Represents ideal dynamic performance.
- Example: Potentiometer, ideal spring etc.



(a) Step input (b) Harmonic input

T855

Zero-order instrument's response for step and harmonic inputs (for $\Bbbk = 0.75$).

First Order System

$$a_1\frac{dx}{dt} + a_0x = f(t) :\Longrightarrow \tau\,\frac{dx}{dt} + x = \Bbbk\, f(t)$$

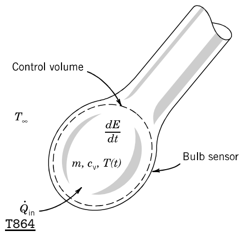
$\Bbbk$	$\equiv 1/a_o$	$\equiv$	static sensitivity,
τ	$\equiv a_1/a_o$	$\equiv$	time-constant.

$a_o \iff$ dissipation (electric or thermal resistance).
$a_1 \iff$ storage (electric or thermal capacitance).

↬ Example: Thermometer, capacitor etc.

- The time constant, τ has the dimension of time, while the *static sensitivity* has the dimension of output divided by input.
- When $\tau \to 0$: the effect of the derivative terms becomes negligible and the governing equation approaches to that of a zero-order system.

- Consider a thermocouple initially at temperature, T is suddenly exposed to an environment at T_∞.



- $h \equiv$ convective heat transfer coefficient,
- $A \equiv$ heat transfer surface area,
- $m \equiv$ mass of mercury + bulb,
- $C \equiv$ specific heat of mercury + bulb.

$$\dot{Q}_{in} = hA\,[T_\infty - T(t)] = mC\frac{dT(t)}{dt} :\Longrightarrow \tau\frac{dT(t)}{dt} + T(t) = T_\infty$$

- Time constant, $\tau \equiv \frac{mC}{hA}$
- Static sensitivity, $\mathbb{k} = 1.0$
- $m \uparrow C \uparrow \;\; h \downarrow A \downarrow \Longrightarrow \tau \uparrow$
- Instruments with small $\tau \rightsquigarrow$ good dynamic response.

Response of a 1^{st} Order System: Step Input

$$x = x_o,\; f = 0 : t = 0; \qquad f(t) = A : t > 0$$

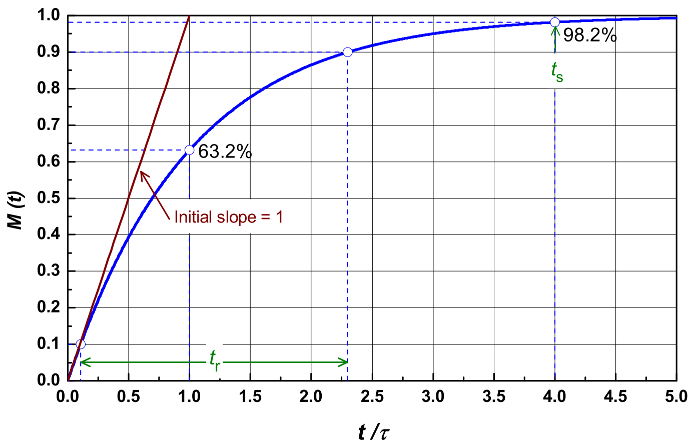
$$\tau\frac{dx}{dt} + x = \mathbb{k}\, f(t)$$

$$\Longrightarrow x(t) = \underbrace{(x_o - A\mathbb{k})\exp(-t/\tau)}_{\text{transient response}} + \underbrace{A\mathbb{k}}_{\text{steady-state response}}$$

- $x(t \to \infty) = A\mathbb{k} = x_\infty \Longleftarrow$ Steady State Response
- Error, $e_m = x_\infty - x(t) = (x_\infty - x_o)e^{-t/\tau}$
- Non-dimensional Error, $e_m/(x_\infty - x_o) = e^{-t/\tau}$

. . . contd

- Non-dimensional response, $M(t) = \frac{x(t)-x_o}{x_\infty - x_o} = 1.0 - \exp(-t/\tau)$



T856

. . . contd.

- Time Constant, τ - time required to complete 63.2% of the process.
- Rise Time, t_r - time required to achieve response from 10% to 90% of final value.
  ↬ For first order system, $t_r = 2.31\tau - 0.11\tau = 2.2\tau$.
- Settling Time, t_s - the time for the response to reach, and stay within 2% of its final value.
  ↬ For first order system, $t_s = 4\tau$.
- Process is assumed to be completed when $t \geq 5\tau$.
- Faster response is associated with shorter τ.

Response of a 1^{st} Order System: Harmonic Input

If the governing equation for first-order system is solved for harmonic input and $x|_{t=0} = 0$, the solution is:

$$\frac{x(t)}{A\mathbb{k}} = \underbrace{\frac{\omega\tau}{1+(\omega\tau)^2}\exp(-t/\tau)}_{\text{transient response}} + \underbrace{\frac{1}{\sqrt{1+(\omega\tau)^2}}\sin(\omega t+\phi)}_{\text{steady-state response}}$$

where, $\phi \equiv \tan^{-1}(-\omega\tau) \equiv$ *phase lag*. Hence, *time delay*, Δt, is related to phase lag as:

$$\Delta t = \frac{\phi}{\omega}$$

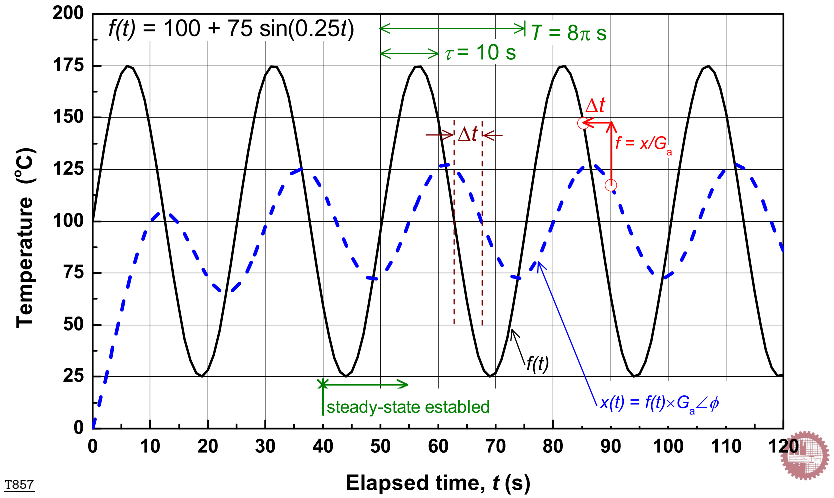
For $\omega\tau >> 1$, response is attenuated and time/phase is lagged from input, and for $\omega\tau << 1$, the transient response becomes very small and response follows the input with small attenuation and time/phase lag.

- Ideal response (without attenuation and phase lag) is obtained when the system time constant, τ is significantly smaller than the forcing element period, $T \equiv 2\pi/\omega$.
- As $t \to \infty$, the steady-state solution:

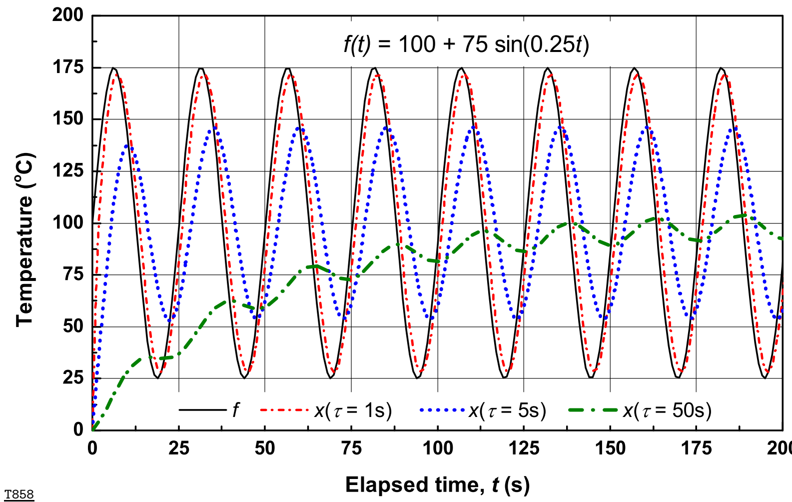
$$x(t)|_s = \frac{A\mathbb{k}}{\sqrt{1+(\omega\tau)^2}}\sin(\omega t+\phi) = G_a f(t)\angle\phi$$

  Hence, $G_a \equiv \mathbb{k}/\sqrt{1+(\omega\tau)^2} \equiv$ *steady-state gain.*
- The attenuated steady-state response is also a sine wave with a frequency equal to the input signal frequency, ω, and it lags behind the input by phase angle, ϕ.

Thermometer ($\tau = 10$s), initially at 0°C
($\omega = 0.25,\ T = 8\pi,\ G_a = 0.37$).



T857



T858

Effects of time constant on system response.

. . . contd.

Unit	τ [s]	τ/T	ϕ [deg]	Δt [s]	G_a
01	01	0.04	-14.0	-0.98	0.97
02	05	0.2	-51.3	-3.58	0.62
03	50	2.0	-85.4	-5.96	0.08

- Response to harmonic input is
  - at same frequency,
  - with a phase shift (time lag), and
  - reduced amplitude.
- The larger the time constant, the greater the phase lag & amplitude decrease (attenuation).

Transfer Function (TF)

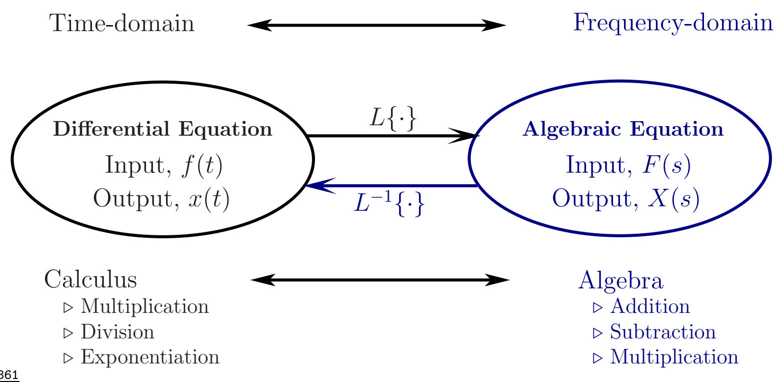
- Transfer function of a linear system, $G(s)$, is defined as the ratio of the Laplace transform (LT) of the output variable, $X(s) \equiv \mathcal{L}\{x(t)\}$, to the LT of the input variable, $F(s) \equiv \mathcal{L}\{f(t)\}$, *with all the initial conditions are assumed to be zero.* Hence,

$$G(s) \equiv \frac{X(s)}{F(s)}$$

∘ The Laplace operator, $s \equiv \sigma + j\omega$, is a complex variable. For steady-state sinusoidal input, $\sigma = 0$, and system response can be evaluated by setting $s = j\omega$.

- ∘ Amplitude gain, $G_a(\omega) \equiv |G(j\omega)|$
- ∘ Phase lag, $\phi(\omega) \equiv \angle G(j\omega)$

$$F(s) \longrightarrow \boxed{G(s)} \longrightarrow X(s) \quad :\Longrightarrow \quad x(t) = f(t) \times G_a \angle \phi$$



T861

Important Laplace Transform Pairs

$f(t)$	$F(s)$
$\delta(t)$	1
t^n	$\frac{n}{s^{n+1}}$
Step function, A	A/s
e^{-at}	$\frac{1}{s+a}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
$\frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin \omega_n \sqrt{1-\zeta^2}\, t, \ \zeta < 1$	$\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$

Bode Diagram

Bode diagram is a pair of graphs which consists of two plots:

1. Logarithmic gain, $L(\omega) \equiv 20\log_{10} G_a(\omega)$ vs. $\log_{10}(\omega)$, and
2. Phase angle, $\phi(\omega)$ vs. $\log_{10}(\omega)$

The vertical scale of the amplitude Bode diagram is in decibels (dB), where a non-dimensional frequency parameter such as frequency ratio, (ω/ω_n), is often used on the horizontal axis.

TF of a 1^{st} Order System

$$\tau \frac{dx}{dt} + x = \Bbbk\, f(t)$$

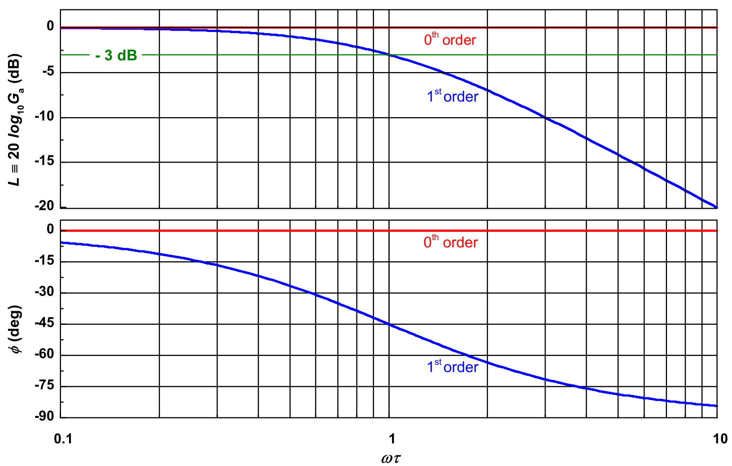
- $\frac{d^n x}{dt^n} \Longrightarrow s^n X(s), \qquad f(t) \Longrightarrow F(s).$

$$\Rightarrow \tau s X(s) + X(s) = \Bbbk\, F(s)$$

$$F(s) \Longrightarrow \boxed{\frac{\Bbbk}{\tau s+1}} \Longrightarrow X(s)$$

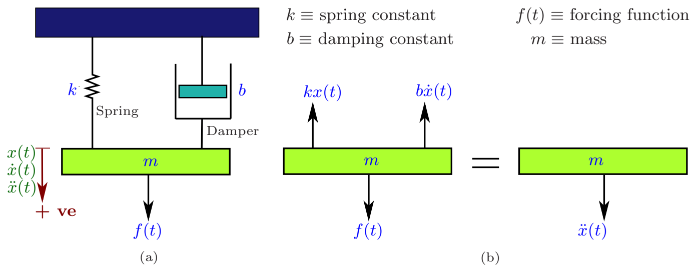
- $s \Longleftarrow j\omega$
- $G_a = |G(j\omega)| = \left|\frac{\Bbbk}{j\omega\tau+1}\right| = \frac{\Bbbk}{\sqrt{1+(\omega\tau)^2}}$
- $\phi = \angle G(j\omega) = \tan^{-1}(-\omega\tau)$

Bode Diagram of a 1^{st} Order System



T862

Second Order System



T851

$$f - kx - c\frac{dx}{dt} = m\frac{d^2x}{dt^2} \Longrightarrow m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = f$$

$$\begin{aligned} \omega_n &\equiv \sqrt{\frac{k}{m}} &&\Longleftrightarrow \text{undamped natural frequency (rad/s)} \\ c_c &\equiv 2\sqrt{mk} &&\Longleftrightarrow \text{critical damping coefficient} \\ \zeta &\equiv c/c_c &&\Longleftrightarrow \text{damping ratio} \end{aligned}$$

TF of a 2^{nd} Order System

$$a_2\frac{d^2x}{dt^2} + a_1\frac{dx}{dt} + a_o x = f(t) :\Longrightarrow \frac{1}{\omega_n^2}\frac{d^2x}{dt^2} + 2\frac{\zeta}{\omega_n}\frac{dx}{dt} + x = \Bbbk\, f(t)$$

where,

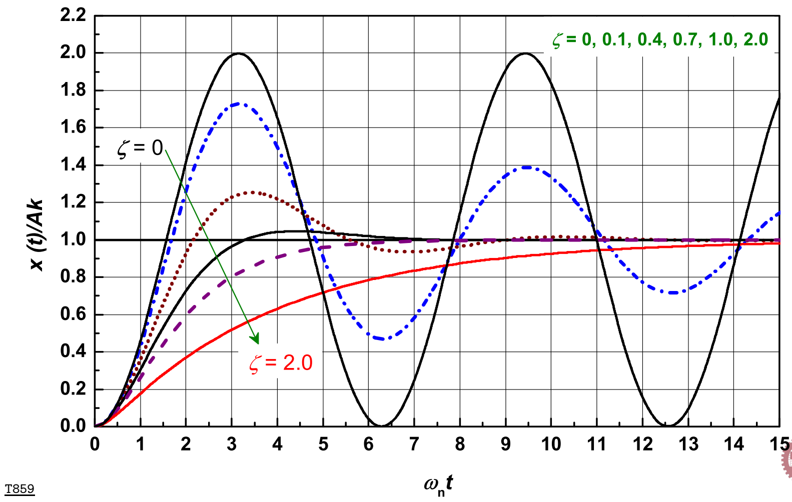
$$\Bbbk \equiv 1/a_o \equiv \text{static sensitivity,}$$

$$\omega_n \equiv \sqrt{\frac{a_o}{a_2}} \equiv \text{undamped natural frequency,}$$

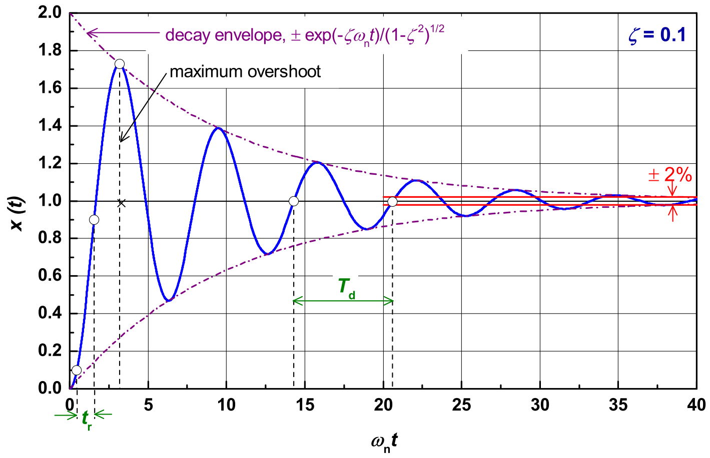
$$\zeta \equiv \frac{a_1}{2\sqrt{a_o a_2}} \equiv \text{dimensionless damping ratio.}$$

- $G(s) = \frac{1/k}{\frac{1}{\omega_n^2}s^2 + 2\frac{\zeta}{\omega_n}s + 1} = \frac{\omega_n^2/k}{s^2 + 2\zeta\omega_n s + \omega_n^2}$
- $G(j\omega) = \frac{1/k}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right] + j\left[2\zeta\frac{\omega}{\omega_n}\right]}$
- $G_a = |G(j\omega)| = \frac{1/k}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + 4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2}}$
- $\phi = \angle G(j\omega) = \tan^{-1}\left[-\frac{2\zeta\frac{\omega}{\omega_n}}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]}\right]$

Response of a 2^{nd} Order System: Step Input



. . . contd.



Second-order under-damped response specifications.

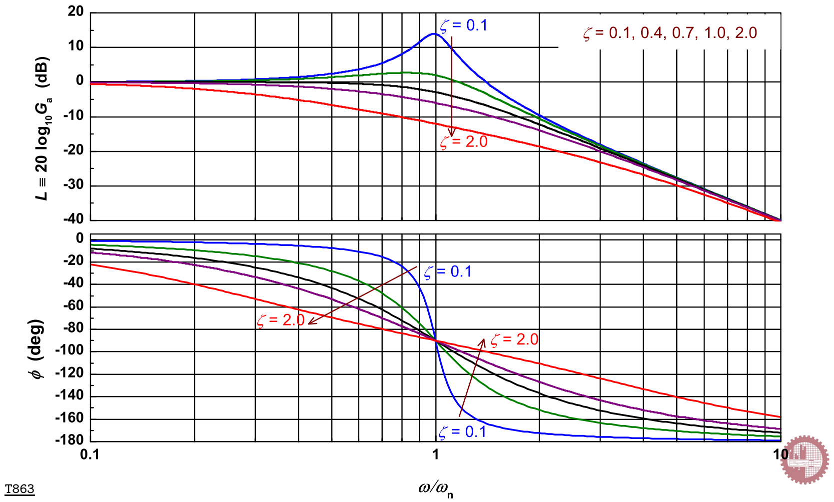
. . . contd.

- Steady state position is obtained after a long period of time.
- Under-damped system ($\zeta < 1$): response overshoots the steady-state value initially, & then eventually decays to the steady-state value. The smaller the value of ζ, the larger the overshoot. The transient response oscillates about the steady-value and occurs with a period, T_d, given by:

$$T_d \equiv \frac{2\pi}{\omega_d} \quad : \quad \omega_d \equiv \omega_n\sqrt{1-\zeta^2}$$

- Critical damping ($\zeta = 1$): an exponential rise occurs to approach the steady-state value without any overshoot.
- Over-damped ($\zeta > 1$): the system approaches the steady-state value without overshoot, but at a slower rate.

Response of a 2^{nd} Order System: Harmonic Input



T863

. . . contd.

- System has a good *linearity* for low damping ratios and up to a frequency ratio of 0.3 since the amplitude gain is very nearly unity ($G_a \simeq 1$).
- For large values of ζ, the amplitude is reduced substantially.
- The phase shift characteristics are a strong function of frequency ratio for all frequencies.
- As a general rule of thumb, the choice of $\zeta = 0.707$ is optimal since it results in the best combination of amplitude linearity and phase linearity over the widest range of frequencies.

Measuring System Response

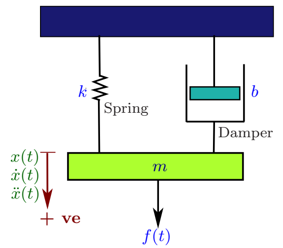
Response is a measure of a system's fidelity to purpose.

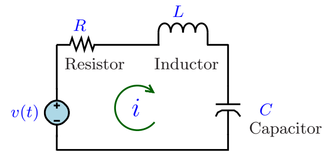
1. Amplitude response:
   - A linear response to various input amplitudes within range.
   - Beyond the linear range, the system is said to be overdriven.
2. Frequency response: is the ability of the system to treat all frequencies the same so that the gain amplitude remains the same over the frequency range desired.
3. Phase response: is important for complex waveforms. Lack of good response may result in severe distortion.
4. Delay, Rise time, Slew rate:
   - Delay or rise time is required to respond to an input quantity.
   - *Slew rate* is the maximum applicable rate of change.

Spring-mass-damper system & analogous RLC circuit

$f(t) \equiv$ forcing function (N)
$m \equiv$ mass (kg)
$k \equiv$ spring constant (N/m)
$b \equiv$ damping constant (N.s/m)
$x \equiv$ displacement (m)
$\dot{x} \equiv dx/dt \equiv$ velocity (m/s)

$v(t) \equiv$ applied voltage (V)
$L \equiv$ inductance (H)
$C \equiv$ capacitance (F)
$R \equiv$ resistance (Ω)
$q \equiv$ charge (C)
$i \equiv dq/dt \equiv$ current (A)





$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = v(t)$$

$L \sim m, R \sim b, \frac{1}{C} \sim k, v \sim f$

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = f(t) \implies \frac{1}{\omega_n^2}\frac{d^2x}{dt^2} + 2\frac{\zeta}{\omega_n}\frac{dx}{dt} + x = \kappa f(t)$$

T852

Bibliography

1. Haq, M.Z., "Measurement: System, Uncertainty and Response". In *Applied Measurement Systems*, 1-22, Edited by Md. Zahurul Haq, InTech, 2011.
2. Figliola, R.S. & Beasley, D.E., *Theory & Design for Mechanical Measurements*, J. Wiley & Sons, Inc, 2011.
3. Halman, J.P., *Experimental Methods for Engineers*, McGraw-Hill, Inc, 2011.
4. Beckwith, T.G., Marangoni, R.D. & Lienhard, J.H., *Mechanical Measurements*, Addison Wesley, Inc, 2006.