Measurement Error and Uncertainty

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ME 361: Instrumentation and Measuremen

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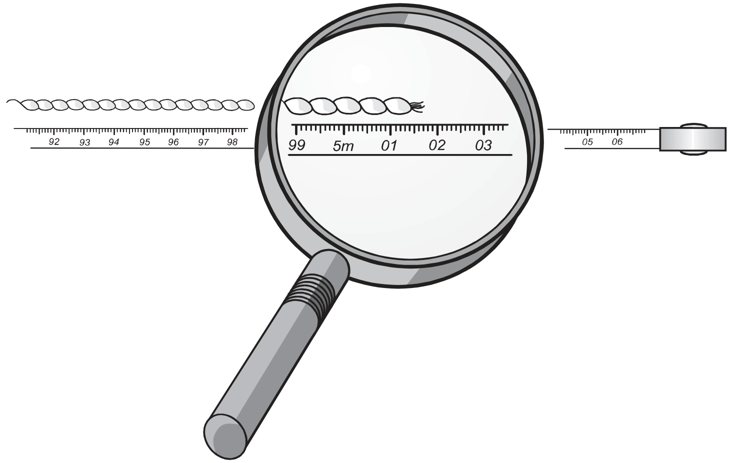

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How long is a piece of string?

Precision Example: How long is the pencil?

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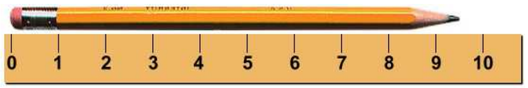
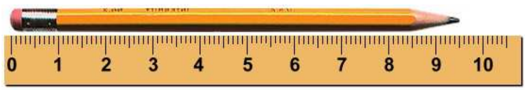


The best you can say is 'about 9 centimetres'

e556



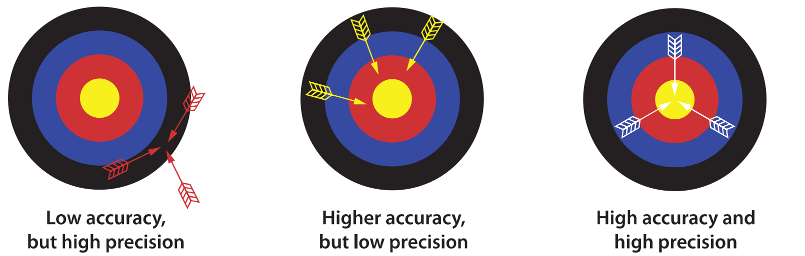
The best you can say is 'about 9.5 centimetres'. This second measurement is more precise, because you used a smaller unit to measure with.

- It is impossible to make a perfectly precise measurement.
- Accuracy can be improved up to but not beyond the precision of the instrument by calibration.

Basic Terminology

- Error is the difference between the measured value and the true value of the thing being measured.
- True value is the value that would be obtained by a theoretically perfect measurement.
- Uncertainty is the quantification of the doubt about the measurement result and tells us something about its quality.
- Accuracy a qualitative term that describes how close a set of measurements are to the actual (true) value.
- Precision describes the spread of these measurements when repeated - a measurement that has high precision has good repeatability.

- Repeatability is the closeness of agreement between repeated measurements of the same thing, carried out in the same place, by the same person, on the same equipment, in the same way, at similar times.
- Reproducibility is the closeness of agreement between measurements of the same thing carried out in different circumstances, e.g. by a different person,or a different method, or at a different time.
- Range: A calibration applies known inputs ranging from the minimum to the maximum values for which the measurement system is to be used. These limits define the operating range of the system.
- Resolution represents the smallest increment in the measured value that can be discerned. It is quantified by the smallest scale increment or least count (least significant digit) of the output readout indicator.
- Error is not the same as mistakes.



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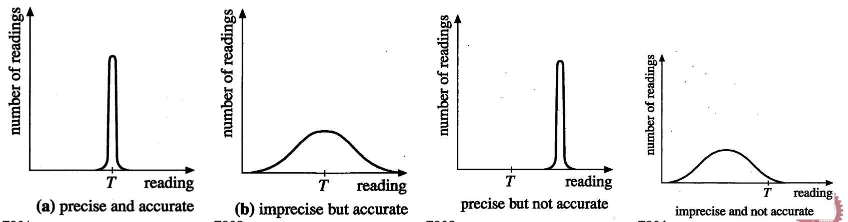

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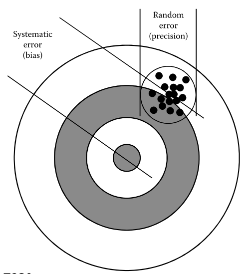

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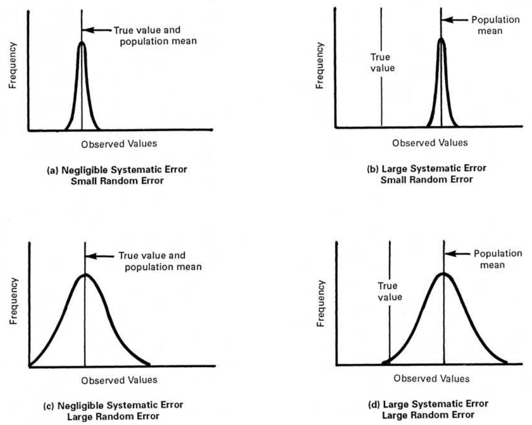

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Components of Total Error: Random error, ϵ

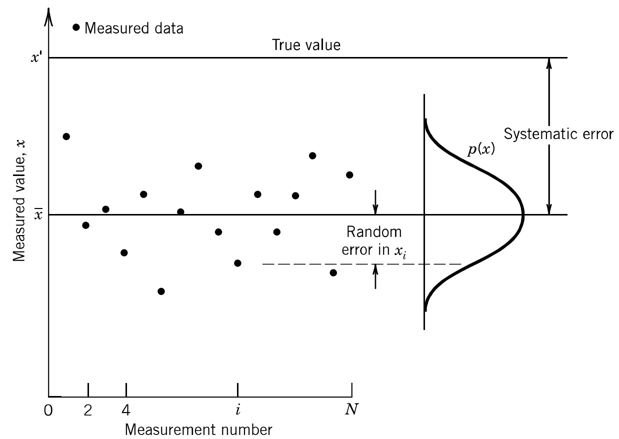
It is the portion of the measurement error that varies randomly in repeated measurements throughout the conduct of a test. Random errors may arise from uncontrolled test conditions and non-repeatability in the measurement system, measurement methods, environmental conditions, data reduction techniques, etc.

- Distinguished by their lack of consistency. Usually (not always) follow a certain statistical distribution.
- In many instances very difficult to distinguish from bias errors.
  - Error stemming from environmental variations
  - Error resulting from variations in definition
  - Error derived from instruments insufficient sensitivity.

Components of Total Error: Systematic error, β

It is the component of measurement error that in replicate measurements remains constant or varies in a particular manner.

- Insidious in nature, exist unnoticed unless deliberately searched.
- Repeated readings to be in error by roughly the same amount.
    - Calibration errors
    - Certain consistently recurring human error
    - Uncorrected loading error
    - Limitations of system resolution
- Four methodologies to reduce systematic errors:
    1. Calibration, by checking the output(s) for known input(s).
    2. Concomitant method, by using different methods of estimating the same thing and comparing the results.
    3. Inter-laboratory comparison, by comparing the results of similar measurements.
    4. Experience.



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Distribution of errors on repeated measurements. Random errors are identified and quantified through repeated measurements while trying to keep the conditions constant and by statistical analysis of the results.

Error Sources

Error group	Error source
1. Calibration error	Standard or reference value errors Instrument or system errors Calibration process errors Calibration curve fit
2. Loading error	Interaction between the instrument and test media Interaction between test article and test facility
3. Data-acquisition error	Measurement system operating conditions Sensor-transducer stage (instrument error) Signal conditioning stage (instrument error) Output stage (instrument error) Process operating conditions Sensor installation effects Environmental effects Spatial variation error Temporal variation error
4. Data-reduction error	Calibration curve fit Truncation error

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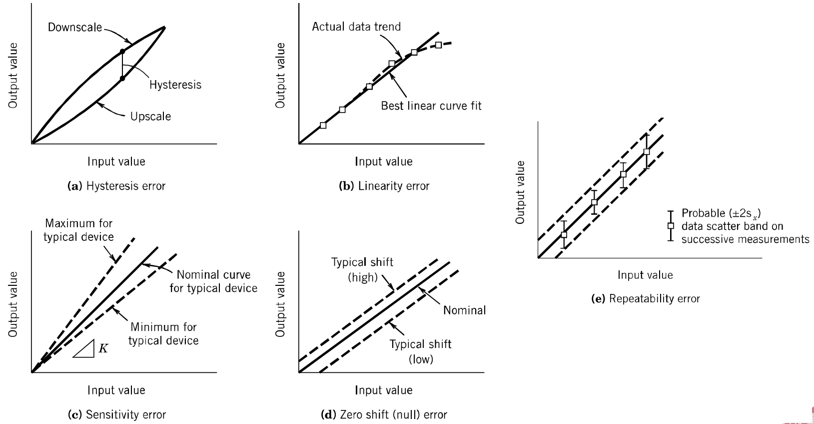
Examples of Elements of Instrumental Errors



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Uncertainty Analysis

- Uncertainty is based on a careful specification of the uncertainties in the various primary experimental measurements. For example, a particular stick is 200 cm long with an uncertainty of ± 1 cm.
- If result $R = R(x_1, x_2, \cdots, x_n)$ is a given function of independent variables $x_1, x_2, \cdots, x_n$; and $w_1, w_2, \cdots, w_n$ are the associated uncertainties, then uncertainty of of the result w_R is given by:

$$w_R = \sqrt{\left(\frac{\partial R}{\partial x_1} w_1\right)^2 + \left(\frac{\partial R}{\partial x_2} w_2\right)^2 + \cdots + \left(\frac{\partial R}{\partial x_n} w_n\right)^2}$$

- The best estimate of the true mean value, R_t, would be stated as:

$$R_t = \overline{R} \pm w_R \quad (P\%)$$

where the sample mean, $\overline{R}$ is found from

$$\overline{R} = R(\overline{x}_1, \overline{x}_2, \cdots, \overline{x}_N)$$

Example: Uncertainty in Power Estimation

Consider the calculation of electrical power, $P = EI$

$$E = 100 \text{ V} \pm 5 \text{ V} \qquad I = 10 \text{ A} \pm 0.1 \text{ A}$$

- Nominal power, $\overline{P} = \overline{E} \times \overline{I} = 100 \times 10 = 1000$ W
- $P_{max} = E_{max} I_{max} = 105 \times 10.1 = 1060.5$ W
- $P_{min} = E_{min} I_{min} = 95 \times 9.9 = 940.5$ W
- $\frac{\partial P}{\partial E} = I = 10$ A, $\quad \omega_E = 5$ V
- $\frac{\partial P}{\partial I} = E = 100$ V, $\quad \omega_I = 0.1$ A

$\Rightarrow$ $\omega_P = \sqrt{(10 \times 5)^2 + (100 \times 0.1)^2} = \sqrt{2500 + 100} = 51.0 \text{ W} = 5.1\%$

$\Longrightarrow$ Estimated power, $P = 1000 \; W \pm 5.1\%$

◇ *Very little is gained by trying to reduce the 'small' uncertainties. Any improvement in result should be achieved by improving the instrumentation or technique connected with the relatively large uncertainties.*

- Uncertainties for Additive Functions: If R is expressed as:

$$R = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = \sum a_i x_i$$

$$\frac{\partial R}{\partial x_i} = a_i$$

$$\Rightarrow w_R = \left[\sum \left(\frac{\partial R}{\partial x_i} w_{x,i}\right)^2\right]^{1/2} = \left[\sum (a_i w_{x,i})^2\right]^{1/2}$$

- Uncertainties for Product Functions: If R is expressed as:

$$R = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$$

$$\frac{\partial R}{\partial x_i} = x_1^{a_1} x_2^{a_2} \cdots (a_i x_i^{a_i - 1}) \cdots x_n^{a_n} \qquad \Rightarrow \frac{1}{R}\frac{\partial R}{\partial x_i} = \frac{a_i}{x_i}$$

$$\Rightarrow \frac{w_R}{R} = \left[\sum \left(\frac{a_i w_{x,i}}{x_i}\right)^2\right]^{1/2}$$

Holman Ex. 3.1: ▷ Resistance of a copper wire is: $R = R_0[1 + \alpha(T - 20)]$, where $R_0 = 6\Omega \pm 0.3\%$ is the resistance at 20°C, $\alpha = 0.004^o C^{-1} \pm 1\%$ is the temperature coefficient of resistance, and the temperature of the wire is $T = 30 \pm 1$°C. Calculate the resistance of the wire and its uncertainty.

- $\overline{R} = (6)[1 + 0.004(30 - 20)] = 6.24\Omega$
- $\frac{\partial R}{\partial R_0} = 1 + \alpha(T - 20) = 1.04$
- $\frac{\partial R}{\partial \alpha} = R_0(T - 20) = 60.0$
- $\frac{\partial R}{\partial T} = R_0 \alpha = 0.024$
- $w_{R_0} = (6)(0.3/100) = 0.018\Omega$
- $w_\alpha = (0.004)(1/100) = 4 \times 10^{-5} \; ^oC^{-1}$
- $w_T = 1^o C$

$\Rightarrow$ $w_R = [(1.04 \times 0.018)^2 + (60 \times 4 \times 10^{-5})^2 + (0.024 \times 1)^2]^{1/2} = 0.0305\Omega$

Statistical Data Analysis

- Arithmetic mean, $x_m = \dfrac{1}{N}\sum_{i=1}^{N} x_i$
- Standard deviation, $\sigma = \left[\dfrac{1}{N}\sum_{i=1}^{N}(x_i - x_m)^2\right]^{1/2}$
- Sample standard deviation, $\sigma = \left[\dfrac{1}{N-1}\sum_{i=1}^{N}(x_i - x_m)^2\right]^{1/2}$
- Standard deviation of mean, $\sigma_m = \dfrac{\sigma}{\sqrt{N}}$

Confidence Interval and Level of Significance

- Confidence interval expresses the probability that the mean value will lie within a certain number of σ values and is given by z. Thus,

$$\overline{x}_t = \overline{x} \pm z\sigma = \overline{x} \pm \Delta \qquad (\%\ \textit{confidence level}\ )$$

- For small data samples, $\Delta = \frac{z\sigma}{\sqrt{N}}$
- Level of significance is 100 percent minus confidence level.

Confidence Interval	Confidence Level, %	Level of Significance, %
3.30	99.9	0.1
3.0	99.7	0.3
2.57	99.0	1.0
2.0	95.4	4.6
1.96	95.0	5.0
1.65	90.0	10.0
1.0	68.3	31.7

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$\Rightarrow$ *mean will lie within $\pm 2.57\sigma$ with less than 1% error (confidence level is 99%).*

Holman Ex. 3.11: $\triangleright$ A certain steel bar is measured with a device which has a known precision of ± 0.5 mm when a large number of measurements is taken. How many measurements are necessary to establish the mean length $\bar{x}$ with a 5 percent level of significance such that $\bar{x} = \bar{x} \pm 0.2$ mm.

- For 5% level of significance, $z = 1.96$.
- $\Delta = \frac{z\sigma}{\sqrt{N}} = \frac{(1.96)(0.5mm)}{\sqrt{N}} = 0.2mm$

$\Rightarrow$ $N = 24.01$

$\Longrightarrow$ So, for 25 measurements or more, we could state with a confidence level of 95 percent that the population mean value will be within ± 0.2 mm of the sample mean value.

Student's t-Distribution

True mean value, x_t, based on the finite data-set can be estimated using:

$$x_t = x_m \pm t_{(\nu,P)}\sigma_m = x_m \pm \Delta \quad (P\%)$$

- $\Delta = t_{(\nu,P)}\sigma_m = t_{(\nu,P)}\frac{\sigma}{\sqrt{N}}$
- $\nu \equiv$ degrees of freedom, $\nu = N - 1$
- $\%P \equiv$ given confidence level (probability)

$\Longrightarrow$

$$\Delta = \begin{cases} z\frac{\sigma}{\sqrt{N}} & \text{for large sample size, } N \geq 20 \\ t_{(\nu,P)}\frac{\sigma}{\sqrt{N}} & \text{for small sample size,} N < 20 \end{cases}$$

Values of Student's t-distribution

ν	t_{50}	t_{90}	t_{95}	t_{99}
1	1.000	6.314	12.706	63.657
2	0.816	2.920	4.303	9.925
3	0.765	2.353	3.182	5.841
4	0.741	2.132	2.770	4.604
5	0.727	2.015	2.571	4.032
6	0.718	1.943	2.447	3.707
7	0.711	1.895	2.365	3.499
8	0.706	1.860	2.306	3.355
9	0.703	1.833	2.262	3.250
10	0.700	1.812	2.228	3.169
11	0.697	1.796	2.201	3.106
12	0.695	1.782	2.179	3.055
13	0.694	1.771	2.160	3.012
14	0.692	1.761	2.145	2.977
15	0.691	1.753	2.131	2.947
16	0.690	1.746	2.120	2.921
17	0.689	1.740	2.110	2.898
18	0.688	1.734	2.101	2.878
19	0.688	1.729	2.093	2.861
20	0.687	1.725	2.086	2.845
21	0.686	1.721	2.080	2.831
30	0.683	1.697	2.042	2.750
40	0.681	1.684	2.021	2.704
50	0.680	1.679	2.010	2.679
60	0.679	1.671	2.000	2.660
∞	0.674	1.645	1.960	2.576

T936

Holman Ex. 3.22: ▷ Ten observations of a voltage are made with $\bar{e} = 15$ V and $\sigma = \pm 0.1$ V. Determine the 5 and 1 percent significance levels.

- $N = 10 \rightarrow \nu = 9$
- For 5% significance, probability is 95%, and 1% significance, P is 99%.
- For 5% significance: from table, $t_{9,95} = 2.262$
- $\Delta = \frac{(2.262)(0.1)}{\sqrt{10}} = 0.0715$ V, for 95% probability.

⇒ For 95% confidence level, $\bar{e} = 15\ V \pm 0.0715\ V$

- For 1% significance: from table, $t_{9,99} = 3.250$,
- $\Delta = \frac{(3.250)(0.1)}{\sqrt{10}} = 0.1028$ V, for 99%. probability.

⇒ For 99% confidence level, $\bar{e} = 15\ V \pm 0.1028\ V$

Holman Ex. 3.24: ▷ Ten measurements are made of the thickness of a metal plate which give 3.61, 3.62, 3.60, 3.63, 3.61, 3.62, 3.60, 3.62, 3.64, and 3.62 mm. Determine the mean value and the tolerance limits for a 90% confidence level.

- Arithmetic mean, $x_m = \frac{1}{N}\sum_{i=1}^{N} x_i = \frac{1}{10}(36.17) = 3.617$ mm.
- Sample standard deviation, $\sigma = \left[\frac{1}{N-1}\sum_{i=1}^{N}(x_i - x_m)^2\right]^{1/2} = 0.0125$ mm.
- With $N = 10 \rightarrow \nu = 9$, and 90% confidence level, $t_{9,90} = 1.833$
- $\Delta = \frac{t_{(\nu,P)}\sigma}{\sqrt{N}} = 0.00726$ mm.

⟹ $x = 3.617 \pm 0.00726$ mm (90% confidence level)

Holman Ex. 3.25: ▷ If the results of the measurements of Example 3.24 are stated as $x_m = 3.617 mm \pm 0.01 mm$, what confidence level should be assigned to this statement?

- From Ex. 3.24, $\sigma = 0.0125$ mm, $\nu = 9$
- $\Delta = \frac{t_{(\nu,P)}\sigma}{\sqrt{N}} = \frac{t_{(9,P)}\sigma}{\sqrt{10}} = 0.01$ mm $\rightarrow t_{9,P} = 2.53$
- By interpolation, $P = 96\%$

⇒ confidence level is 96%.

Figliola Ex. 4.4: ▷ (changed data-set)

9.78	10.29	9.68	10.30	9.69	9.63	10.35	9.88	10.29	9.72
9.85	10.25	9.75	10.24	9.89	10.31	9.94	10.35	10.27	9.65

- Sample mean, $x_m = \frac{1}{N}\sum_{j=1}^{N} x_j = 10.0$
- Sample standard deviation, $\sigma = \sqrt{\frac{1}{N-1}\sum_{j=1}^{N}(x_j - \overline{x})^2} = 0.28$
- The true mean value, x_t, is estimated by the sample mean value, $\overline{x}$ and the standard deviation of the mean, $\sigma_m = \sigma/\sqrt{N} = 0.0626$:

$$\begin{aligned} x_t &= 10.00 \pm (1.729 \times 0.0626) = 10.00 \pm 0.11 \quad (90\%) \\ &= 10.00 \pm (2.093 \times 0.0626) = 10.00 \pm 0.13 \quad (95\%) \end{aligned}$$

- If a 21^{st} data points are to be taken:

$$\begin{aligned} x_{j=21} &= 10.00 \pm (1.729 \times 0.28) = 10.00 \pm 0.48 \quad (90\%) \\ &= 10.00 \pm (2.093 \times 0.28) = 10.00 \pm 0.59 \quad (95\%) \end{aligned}$$

So, with 95% probability, 21^{st} value will be between 9.41 and 10.59.

Significant Figures/Digits

- Significant figures in a number are those that are known with certainty. A measured value represented by N digits consists of $N - 1$ significant digits that are certain and 1 digit that is estimated. This problem is known as *last digit bobble*.
- The number of significant figures is not improved by combining the numbers with other numbers.
- Example: $15.6\hat{5} \times 0.02\hat{5}$, $\hat{}$ means not-significant.
  Nominal value $= 15.65 \times 0.025 = 0.3\hat{9}\hat{1}\hat{2}\hat{5}$.
  15.65 has 3 & 0.025 has 1 significant digits.
  Result should be 0.4, *rather than* 0.39.
  $15.64 \times 0.024 = 0.37536$, & $15.66 \times 0.026 = 0.40716$.